

Topological wave functions and heat equations

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ABSTRACT: It is generally known that the holomorphic anomaly equations in topological string theory reflect the quantum mechanical nature of the topological string partition function. We present two new results which make this assertion more precise: (i) we give a new, purely holomorphic version of the holomorphic anomaly equations, clarifying their relation to the heat equation satisfied by the Jacobi theta series; (ii) in cases where the moduli space is a Hermitian symmetric tube domain G/K , we show that the general solution of the anomaly equations is a matrix element $\langle \Psi | g | \Omega \rangle$ of the Schrödinger-Weil representation of a Heisenberg extension of G , between an arbitrary state $\langle \Psi |$ and a particular vacuum state $|\Omega\rangle$. Based on these results, we speculate on the existence of a one-parameter generalization of the usual topological amplitude, which in symmetric cases transforms in the smallest unitary representation of the duality group G' in three dimensions, and on its relations to hypermultiplet couplings, nonabelian Donaldson-Thomas theory and black hole degeneracies.

KEYWORDS: Black Holes in String Theory, Topological Strings.

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1. Introduction

On top of its rich mathematical structure, topological string theory offers the most practical way of computing certain higher-derivative corrections to the low energy effective action in $\mathcal{N} = 2$ compactifications of physical string theory. It has played a central role in recent studies of subleading corrections to the Bekenstein-Hawking entropy of BPS black holes [1–3]. A particularly striking development is a direct relation between the degeneracies of black hole micro-states and the topological string partition function, conjectured by Ooguri-Strominger-Vafa (OSV) [4].

The topological string partition function satisfies “holomorphic anomaly” equations derived by Bershadsky-Cecotti-Ooguri-Vafa (BCOV) [5, 6], which describe its dependence on a “background” point of moduli space. These equations are best understood by viewing the topological string partition function as the wave function of a background-independent state in a certain Hilbert space, expressed in a background-dependent basis of coherent states [7]. In particular, as emphasized in [8], for the application to the OSV conjecture one should choose a basis of coherent states corresponding to a “real polarization” of the wave function. The purpose of the present work is to tighten the wave function interpretation of the holomorphic anomaly equations, by recasting these equations in a purely holomorphic way, and representing their solution as a matrix element in an appropriate representation of the continuous global symmetry group of the moduli space, when such a symmetry exists.

To motivate our approach, recall that the wave function interpretation of the holomorphic anomaly equations was originally suggested [7] by analogy with the heat equations

$$\left[i \frac{\partial}{\partial t} - \frac{\partial^2}{\partial y^2} \right] \theta(t, y) = 0, \tag{1.1}$$

$$\frac{\partial}{\partial \bar{t}} \theta(t, y) = 0, \tag{1.2}$$

obeyed by the classical Jacobi theta series $\theta(t, y)$. However, the anomaly equations as written in [6] are not quite identical to (1.1), (1.2). In particular, the topological string partition function Ψ_{BCOV} is neither purely holomorphic nor purely antiholomorphic. So the precise analogy between Ψ_{BCOV} and $\theta(t, x)$ has remained slightly obscure.

In section 2 of this paper, we draw a precise analogy between the topological string partition function Ψ_{BCOV} (or rather, its variant $\Psi_{\text{V}}(t, \bar{t}, x)$ as defined in [8]), and the complex conjugate of the “canonical” Jacobi theta function $\Theta(t, \bar{t}, \bar{x})$,

$$\Psi_{\text{V}}^*(t, \bar{t}, \bar{x}) \sim \Theta(t, \bar{t}, \bar{x}), \tag{1.3}$$

where $\Theta(t, \bar{t}, \bar{x})$ is a non-holomorphic modular function which differs from the classical holomorphic $\theta(t, y)$ by a simple transformation,

$$\Theta(t, \bar{t}, \bar{x}) = \sqrt{\text{Im } t} \exp\left(\frac{y^2}{\text{Im } t}\right) \theta(t, y), \tag{1.4}$$

$$y = (\text{Im } t)\bar{x}. \tag{1.5}$$

Namely, by transformations of Ψ_{V}^* similar to (1.4), (1.5), described in section 2.6, we produce a modified partition function Ψ_{hol} , which is purely holomorphic and obeys a heat equation (2.69) closely analogous to (1.1). These transformations, valid in complete generality, provide a more concrete footing for the widely accepted idea that the anomaly equations of the topological string partition function should be analogous to the heat equation of a theta function. Along the way we also clarify a few details which have not appeared explicitly in the literature, in particular the relation between the one-loop holomorphic anomaly and the metaplectic correction to geometric quantization.

Importantly, our transformation from Ψ_V^* to Ψ_{hol} introduces a dependence on a choice of symplectic basis, which is not invariant under large biholomorphisms of the Calabi-Yau threefold; consequently, the transformation of the holomorphic partition function Ψ_{hol} under large biholomorphisms is more complicated than that of the original Ψ_V , as we discuss in section 2.8. This parallels the fact that the holomorphic $\theta(t, z)$ has a more complicated transformation under $SL(2, \mathbb{Z})$ than does the non-holomorphic $\exp\left(\frac{z^2}{\text{Im}t}\right)\theta(t, z)$.

In section 3, we specialize the discussion to cases where the moduli space is a symmetric special Kähler manifold G/K . While such cases are very special (in particular they require the absence of genus 0 worldsheet instanton corrections), they allow us to develop a second aspect of the analogy between Ψ and $\theta(t, z)$, representing the topological amplitude as a matrix element in a unitary representation of an extension of G . To appreciate this point, recall the standard interpretation of $\theta(t, z)$ in representation theory [9, 10]: One considers the semi-direct product $\tilde{G} = SL(2, \mathbb{R}) \ltimes H$, where H is the three-dimensional Heisenberg algebra $[\hat{q}, \hat{p}] = Z$, and $G = SL(2, \mathbb{R})$ acts on H/Z in the fundamental representation. Both G and H act naturally on the Hilbert space obtained by quantizing \mathbb{R}^2 , which we call the ‘‘Schrödinger-Weil representation’’ of \tilde{G} . Then $\theta(t, z)$ may be written as a matrix element in this representation,

$$\theta(t, y) = \langle \Psi | e^{y\hat{p}} e^{tT} | \Omega \rangle, \tag{1.6}$$

where T is the positive root generator $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ of $\mathfrak{sl}(2, \mathbb{R})$, $|\Omega\rangle$ is a state annihilated by \hat{q} , and $\langle \Psi |$ is a state invariant under a subgroup of $SL(2, \mathbb{Z})$. In this interpretation the heat equation (1.1) becomes a consequence of an operator identity in the Schrödinger-Weil representation,

$$ZT - \hat{p}^2 = 0, \tag{1.7}$$

if we choose the central generator to act as $Z = i$. Note that the heat equation would hold no matter which state $|\Psi\rangle$ we choose; the most general solution is obtained by choosing an arbitrary $|\Psi\rangle$.

We show that a similar story holds for the topological string partition function, as one would expect from its wave function interpretation: there is a 1-1 correspondence between states $|\Psi\rangle$ in the Schrödinger-Weil representation of an appropriate group $\tilde{G} = G \ltimes H$ and solutions to the anomaly equation (2.69). Given $|\Psi\rangle$, the corresponding solution can be written as a matrix element similar to (1.6), given in (3.23). From this point of view, the holomorphic anomaly equations are consequences of identities (3.20) in the Schrödinger-Weil representation, analogous to (1.7) (but slightly more complicated.)

In section 3.8 we discuss a third aspect of the analogy between Ψ and θ , which may be motivated as follows. Equation (1.7) can be obtained by noting that the non-reductive group $SL(2, \mathbb{R}) \ltimes H$ is naturally obtained as a subgroup of the more familiar $Sp(4, \mathbb{R})$, and the Schrödinger-Weil representation of $SL(2, \mathbb{R}) \ltimes H$ is contained in the metaplectic (Weil) representation of $Sp(4, \mathbb{R})$. From this point of view, (1.7) is just one of a large class of quadratic operator equations which hold in the Weil representation of $Sp(4, \mathbb{R})$. We note that a very similar situation obtains for the topological string: the identities (3.20) involving generators of \tilde{G} come directly from relations in the ‘‘minimal representation’’ of a

larger and more familiar group G' . We also briefly discuss the automorphic version of this relation, which in the $Sp(4, \mathbb{R})$ case gives rise to the notion of Fourier-Jacobi coefficients.

We emphasize that none of our rigorous results bear directly on the problem of *computing* the topological string partition function; they are, rather, a clarification of what kind of object the partition function is. However, they do tie in naturally to some other directions of inquiry which were really the original motivation, and which we discuss in section 4.

In particular, we note that the group G' mentioned above arises physically as the local isometry group of the moduli space upon compactification from $d = 4$ to $d = 3$. The group G' has been proposed as a spectrum generating symmetry group in $d = 4$ supergravity, and its minimal unitary representation studied in connection with black hole degeneracies (see [11] and references therein.) For the relation to the topological string, the most relevant point is that the minimal representation of G' contains a one-parameter family of Schrödinger-Weil representations of \tilde{G} with varying values for the center $Z = i\hbar$, with the representation relevant for the topological string arising at $\hbar = 1$. This suggests the existence of a “generalized topological amplitude”, a one-parameter extension of the usual perturbative topological amplitude, which (in the special cases we consider) would be described by a state in the minimal representation of G' . We conjecture that this generalized topological amplitude could control corrections to hypermultiplet couplings and be related to the nonabelian Donaldson-Thomas theory. We also briefly discuss the appealing possibility that the topological amplitude could literally be a theta function (in the appropriate sense), which would in principle determine the whole nonperturbative partition function up to at most a finite-dimensional ambiguity.

For completeness, in Appendices A and B we discuss a relation between the real-polarized wave function and the “holomorphic ambiguities” introduced by BCOV in solving the anomaly equations, and the form of the wave function in a positive-definite holomorphic polarization.

Holomorphic rewritings of the anomaly equations in some special cases have appeared before in the literature, *e.g.* [12]. The anomaly equations and their quantum-mechanical interpretation have also been reexamined in [13, 14]. Modular properties of the anomaly equations have recently been discussed in [15], and treated more generally in [16], which appeared while this paper was being finalized. Parts of [16] overlap with parts of section 2 of this paper.

2. The anomaly equations revisited

In this section, we discuss various forms of the holomorphic anomaly equations. Most of these have already appeared in the literature; our main purpose is to introduce a new version of the topological string partition function which is purely antiholomorphic, and whose dependence on the antiholomorphic variables is governed by a heat-type equation given in (2.69) below. Along the way, we also clarify some aspects of the more standard presentations of the topological amplitude. Our discussion is couched in the language of

the B model topological string on a Calabi-Yau threefold X , but with suitable changes of notation and wording it would apply equally well to the A model.

2.1 BCOV's holomorphic anomaly equations

Let \mathcal{M} denote the Teichmüller space of complex structures on X , equipped with its Weil-Petersson metric g . Let \mathcal{T} be the holomorphic tangent bundle to \mathcal{M} , and \mathcal{L} the line bundle $H^{3,0}(X, \mathbb{C})$ over \mathcal{M} .

In terms of local complex coordinates on \mathcal{M} , a point $t \in \mathcal{M}$ may be written $(t^i, \bar{t}^{\bar{i}})$, with $i = 1, \dots, n_v$, where $n_v = h^{2,1}(X)$. We also work in a local trivialization of the line bundle \mathcal{L} , given by some local holomorphic section $\Omega(t)$. Then let K denote the Kähler potential, defined by

$$K(t, \bar{t}) = -\log \|\Omega\|^2 = -\log i \int_X \Omega \wedge \bar{\Omega}. \tag{2.1}$$

The metric compatible connection in \mathcal{L} is

$$D_i s = (\partial_i - \partial_i K) s. \tag{2.2}$$

Moreover, K also serves as a local Kähler potential for the metric on \mathcal{M} :

$$g_{i\bar{j}} = \partial_i \bar{\partial}_{\bar{j}} K. \tag{2.3}$$

Following [6], we consider the generating function of the genus g correlation functions $C_{i_1 \dots i_n}^{(g)}$ of n chiral fields, in the B model topological string theory on X :

$$W(t^i, \bar{t}^{\bar{i}}; x^i, \lambda) = \sum_{g=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{n!} \lambda^{2g-2} C_{i_1 \dots i_n}^{(g)}(t, \bar{t}) x^{i_1} \dots x^{i_n} + \left(\frac{\chi}{24} - 1\right) \log \lambda. \tag{2.4}$$

This is a formal series which need not converge; throughout this paper we will ignore this issue. We consider x^i as the coordinates of a vector $x^i \frac{\partial}{\partial t^i} \in \mathcal{T}$. Similarly we consider the inverse topological string coupling λ^{-1} as the coordinate of an underlying vector $\lambda^{-1} \Omega \in \mathcal{L}$, and $C_{i_1 \dots i_n}^{(g)}$ as the coordinate expression of an element in $(\mathcal{T}^*)^n \otimes \mathcal{L}^{2g-2}$.

The correlation functions $C^{(g)}$ are given by $C_{i_1 \dots i_n}^{(g)} = 0$ for $2g - 2 + n \leq 0$ (so W does not incorporate the genus 0 and 1 vacuum amplitudes) and

$$C_{i_1 \dots i_n}^{(g)}(t, \bar{t}) = \begin{cases} D_{i_1} \dots D_{i_{n-3}} C_{i_{n-2} i_{n-1} i_n} & \text{for } g = 0, n \geq 3, \\ D_{i_1} \dots D_{i_n} F_1 & \text{for } g = 1, n \geq 1, \\ D_{i_1} \dots D_{i_n} F_g & \text{for } g \geq 2. \end{cases} \tag{2.5}$$

In particular, $C_{ijk} = C_{ijk}^{(0)}$ is the tree-level three-point function, related to the curvature of \mathcal{M} by the special geometry formula¹

$$R_{i\bar{j}k\bar{l}} = g_{i\bar{j}} g_{k\bar{l}} + g_{i\bar{l}} g_{k\bar{j}} - e^{2K} C_{ikm} \bar{C}_{\bar{j}\bar{l}\bar{n}} g^{m\bar{n}}. \tag{2.6}$$

¹Note that our definition of C_{ijk} differs from the one used in most of the supergravity literature by a factor e^K .

The covariant derivatives in (2.5) incorporate the Levi-Civita connection acting on \mathcal{T}^* , given in coordinates by $\Gamma_{ij}^k = g^{k\bar{k}} \partial_i g_{j\bar{k}}$, as well as the connection $-\partial K$ in \mathcal{L} . So, for example,

$$D_i F_g = (\partial_i - (2g - 2) \partial_i K) F_g, \tag{2.7}$$

$$D_i C_j^{(g)} = (\partial_i - (2g - 2) \partial_i K - \Gamma_{ij}^k \partial_k) C_j^{(g)}. \tag{2.8}$$

In [6] the topological string partition function is written

$$\Psi_{\text{BCOV}} = \exp \left[W(t^i, \bar{t}^{\bar{j}}; x^i, \lambda) \right]. \tag{2.9}$$

Because of the last term in (2.4), Ψ_{BCOV} is a section of $\mathcal{L}^{1-\frac{\chi}{24}}$.² The holomorphic anomaly equations can be expressed as two conditions on Ψ_{BCOV} :

$$\left[\frac{\partial}{\partial t^i} - \frac{\lambda^2}{2} e^{2K} \bar{C}_{i\bar{j}\bar{k}} g^{j\bar{j}} g^{k\bar{k}} \frac{\partial^2}{\partial x^j \partial x^k} + g_{i\bar{j}} x^j \left(\lambda \frac{\partial}{\partial \lambda} + x^k \frac{\partial}{\partial x^k} \right) \right] \Psi_{\text{BCOV}} = 0, \tag{2.10}$$

$$\left[\frac{\partial}{\partial t^i} - \Gamma_{ij}^k x^j \frac{\partial}{\partial x^k} + \partial_i K \left(\frac{\chi}{24} - 1 - \lambda \frac{\partial}{\partial \lambda} \right) - \frac{\partial}{\partial x^i} + \partial_i F_1 + \frac{1}{2\lambda^2} C_{ijk} x^j x^k \right] \Psi_{\text{BCOV}} = 0. \tag{2.11}$$

The second equation (2.11) summarizes the relations (2.4), (2.5), while (2.10) reproduces the holomorphic anomaly equations proper, which relate the antiholomorphic derivative of the genus- g correlation function to correlators at lower genera.

2.2 Verlinde’s holomorphic anomaly equations

E. Verlinde’s version of the holomorphic anomaly equations [8] is obtained from (2.10), (2.11) by making two transformations. The first step is to rescale $x_{\text{new}}^i = \lambda^{-1} x_{\text{old}}^i$, so that the first term in (2.4) vanishes at $\lambda = 0$. After this rescaling x^i are coordinates in $\mathcal{T} \otimes \mathcal{L}$, so altogether $(t^i, \bar{t}^{\bar{j}}, x^i, \lambda^{-1})$ are coordinates on the total space of $\mathcal{L} \oplus (\mathcal{T} \otimes \mathcal{L})$ over \mathcal{M} ; we call this total space $\widetilde{\mathcal{M}}$. Second, note that the anomaly equation for the one-loop vacuum amplitude [5],

$$\frac{\partial}{\partial t^i} \frac{\partial}{\partial \bar{t}^{\bar{j}}} F_1 = \frac{1}{2} e^{2K} C_{ikl} \bar{C}_{j\bar{k}\bar{l}} g^{k\bar{k}} g^{l\bar{l}} - \left(\frac{\chi}{24} - 1 \right) g_{i\bar{j}}, \tag{2.12}$$

(which is contained in (2.10) at leading order in λ), is solved locally by³

$$F_1 = -\frac{1}{2} \log |g| + \left(\frac{n_v + 1}{2} - \frac{\chi}{24} + 1 \right) K + f_1(t) + \bar{f}_1(\bar{t}). \tag{2.13}$$

Here $f_1(t)$ is a holomorphic function and $|g| = \det(g_{i\bar{j}})$. To patch these local solutions together into the global F_1 , $e^{f_1(t)}$ should be the coordinate expression of a section of $\mathcal{L}^{\frac{\chi}{24}-1-\frac{n_v+1}{2}} \otimes \mathcal{K}_{\mathcal{M}}^{\frac{1}{2}}$, where $\mathcal{K}_{\mathcal{M}}$ is the canonical bundle of \mathcal{M} , locally trivialized by the section $dt^1 \wedge \dots \wedge dt^{n_v}$. Then defining

$$\Psi_V(t, \bar{t}; x, \lambda) = e^{f_1(t)} \Psi_{\text{BCOV}}(t, \bar{t}; \lambda x, \lambda), \tag{2.14}$$

²More precisely, Ψ_{BCOV} is a section of the line bundle obtained by pulling back $\mathcal{L}^{1-\frac{\chi}{24}}$ from \mathcal{M} ; we will abuse language in this way several times.

³This equation corrects a sign error in (4.7) of [8], which affects (4.4) and (4.5).

Ψ_V is a section of $\mathcal{L}^{-\frac{n_v+1}{2}} \otimes \mathcal{K}_{\mathcal{M}}^{\frac{1}{2}}$ over $\widetilde{\mathcal{M}}$, and the anomaly equations for Ψ_V are expressed purely in terms of the special geometry of \mathcal{M} :

$$\left[\frac{\partial}{\partial \bar{t}^i} - \frac{1}{2} e^{2K} \bar{C}_{\bar{i}\bar{j}\bar{k}} g^{j\bar{j}} g^{k\bar{k}} \frac{\partial^2}{\partial x^j \partial x^k} - g_{\bar{i}j} x^j \frac{\partial}{\partial \lambda^{-1}} \right] \Psi_V = 0, \quad (2.15)$$

$$\left[\nabla_{t^i} - \Gamma_{ij}^k x^j \frac{\partial}{\partial x^k} - \frac{1}{2} \partial_{t^i} \log |g| - \lambda^{-1} \frac{\partial}{\partial x^i} + \frac{1}{2} C_{ijk} x^j x^k \right] \Psi_V = 0, \quad (2.16)$$

where

$$\nabla_i = \frac{\partial}{\partial t^i} + \partial_i K \left(x^k \frac{\partial}{\partial x^k} + \lambda^{-1} \frac{\partial}{\partial \lambda^{-1}} + \frac{n_v + 1}{2} \right). \quad (2.17)$$

A crucial observation for what follows [8] is that, provided both Ψ_V and Ψ'_V obey (2.15) and (2.16),

$$\int dx^i d\bar{x}^{\bar{i}} d\lambda^{-1} d\bar{\lambda}^{-1} \sqrt{|g|} e^{-\frac{n_v+1}{2}K} \exp(-e^{-K} x^i g_{i\bar{j}} \bar{x}^{\bar{j}} + e^{-K} \lambda^{-1} \bar{\lambda}^{-1}) \Psi'_V(t, \bar{t}; \bar{x}, \bar{\lambda}) \Psi_V(t, \bar{t}; x, \lambda) \quad (2.18)$$

is a pure number, independent of (t, \bar{t}) . This is proven by formal integration by parts.

2.3 The wave function property and coherent states

The holomorphic anomaly equations were elegantly reinterpreted in [7] as the wave function property of the topological string. In this section we review this reinterpretation.

Recall that Ψ_V is defined on the total space $\widetilde{\mathcal{M}}$ of the holomorphic vector bundle $\mathcal{L} \oplus (\mathcal{T} \otimes \mathcal{L})$ over \mathcal{M} . The fiber at $t \in \mathcal{M}$ of this vector bundle can be identified with the fixed vector space $H^3(X, \mathbb{R})$, equipped with a particular ‘‘Griffiths’’ complex structure \mathcal{J}_t . Explicitly, the correspondence between an element $\gamma \in H^3(X, \mathbb{R})$ and the coordinates (λ^{-1}, x) on $\mathcal{L} \oplus (\mathcal{T} \otimes \mathcal{L})$ is given by the Hodge decomposition of $H^3(X, \mathbb{C})$:

$$\begin{aligned} \gamma &= \lambda^{-1} \Omega + x^i \delta_i \Omega + x^{\bar{i}} \delta_{\bar{i}} \bar{\Omega} + \bar{\lambda}^{-1} \bar{\Omega} \\ &\in H^{3,0} \oplus H^{2,1} \oplus H^{1,2} \oplus H^{0,3} \\ &\simeq \mathcal{L} \oplus (\mathcal{T} \otimes \mathcal{L}) \oplus (\bar{\mathcal{T}} \otimes \bar{\mathcal{L}}) \oplus \bar{\mathcal{L}}. \end{aligned} \quad (2.19)$$

Here we introduced $\delta_i \Omega = (\partial_i + \partial_i K) \Omega$, which is the $(2, 1)$ part of $\partial_i \Omega$. Using this decomposition one can define

$$\mathcal{J}_t = \begin{cases} +i \text{ on } H_+^3 = H^{3,0} \oplus H^{2,1}, \\ -i \text{ on } H_-^3 = H^{0,3} \oplus H^{1,2}. \end{cases} \quad (2.20)$$

Then $H^3(X, \mathbb{R})$ with the complex structure \mathcal{J}_t is identified with the $+i$ -eigenspace H_+^3 . So as a complex vector bundle over \mathcal{M} , $\mathcal{L} \oplus (\mathcal{T} \otimes \mathcal{L})$ is isomorphic to the bundle of $+i$ -eigenspaces H_+^3 , which we write as \mathcal{H}_+^3 .

$H^3(X, \mathbb{R})$ also carries a canonical symplectic structure, inherited from the antisymmetric pairing $(\alpha, \beta) = \int_X \alpha \wedge \beta$ on $\Omega^3(X, \mathbb{R})$: in our (x, λ^{-1}) coordinates it is

$$\omega = i e^{-K} \left(g_{\bar{i}j} dx^i \wedge d\bar{x}^{\bar{j}} - d\lambda^{-1} \wedge d\bar{\lambda}^{-1} \right). \quad (2.21)$$

This ω is compatible with \mathcal{J}_t for every t , so the \mathcal{J}_t give holomorphic polarizations on the fixed space $H^3(X, \mathbb{R})$. Witten's proposal [7], revisited in [8], is that $\Psi_V(t, \bar{t}; x, \lambda)$ at fixed t should be viewed as a wave function obtained by quantizing $H^3(X, \mathbb{R})$ in this holomorphic polarization,⁴ and the anomaly equations (2.15), (2.16) simply implement the infinitesimal Bogoliubov transformation which results from a change of (t, \bar{t}) .

In other words: there is some Hilbert space \mathcal{H} obtained by quantizing $H^3(X, \mathbb{R})$ with the symplectic form ω . In \mathcal{H} there should be a state $|\Psi\rangle$ which contains the background-independent information in the topological string theory. Further, there should be a (t, \bar{t}) -dependent basis of \mathcal{H}^* consisting of “coherent states” ${}_{(t, \bar{t})}\langle x, \lambda|$, such that

$$\Psi_V(t, \bar{t}; x, \lambda) = {}_{(t, \bar{t})}\langle x, \lambda | \Psi \rangle. \quad (2.22)$$

The holomorphic anomaly equations obeyed by Ψ_V then express the variation with (t, \bar{t}) of the states ${}_{(t, \bar{t})}\langle x, \lambda|$; they give no constraint on the state $|\Psi\rangle$.

Let us prove a few facts about the basis of coherent states. By definition they diagonalize the (t, \bar{t}) -dependent annihilation operators $\widehat{\lambda}^{-1}$ and \widehat{x}^i :

$${}_{(t, \bar{t})}\langle x, \lambda | \widehat{\lambda}^{-1}(t, \bar{t}) = {}_{(t, \bar{t})}\langle x, \lambda | \lambda^{-1}, \quad {}_{(t, \bar{t})}\langle x, \lambda | \widehat{x}^i(t, \bar{t}) = {}_{(t, \bar{t})}\langle x, \lambda | x^i. \quad (2.23)$$

We also want to work out the action of the creation operators $\widehat{\lambda}^{-1}(t, \bar{t})$ and $\widehat{x}^i(t, \bar{t})$. These should be the adjoints of the annihilation operators. To determine them explicitly we need to use the formula (2.18), now interpreted as

$$\begin{aligned} \langle \Psi' | \Psi \rangle = \int dx^i d\bar{x}^{\bar{i}} d\lambda^{-1} d\bar{\lambda}^{-1} \sqrt{|g|} e^{-\frac{n_V+1}{2}K} \\ \exp\left(-e^{-K} x^i g_{i\bar{j}} \bar{x}^{\bar{j}} + e^{-K} \lambda^{-1} \bar{\lambda}^{-1}\right) \langle \Psi' | \bar{x}, \bar{\lambda} \rangle \langle x, \lambda | \Psi \rangle, \end{aligned} \quad (2.24)$$

which expresses the “completeness relation” for the basis of coherent states,

$$\int dx^i d\bar{x}^{\bar{i}} d\lambda^{-1} d\bar{\lambda}^{-1} \sqrt{|g|} e^{-\frac{n_V+1}{2}K} \exp\left(-e^{-K} x^i g_{i\bar{j}} \bar{x}^{\bar{j}} + e^{-K} \lambda^{-1} \bar{\lambda}^{-1}\right) |\bar{x}, \bar{\lambda}\rangle \langle x, \lambda| = \mathbf{1}. \quad (2.25)$$

Using (2.25) and integrating by parts, we can show that

$${}_{(t, \bar{t})}\langle x, \lambda | \widehat{\lambda}^{-1} = -e^K \frac{\partial}{\partial \lambda^{-1}} {}_{(t, \bar{t})}\langle x, \lambda|, \quad {}_{(t, \bar{t})}\langle x, \lambda | \widehat{x}^i = e^K g^{i\bar{j}} \frac{\partial}{\partial x^{\bar{j}}} {}_{(t, \bar{t})}\langle x, \lambda|. \quad (2.26)$$

consistently with the commutation rules which follow from (2.21),

$$[\widehat{\lambda}^{-1}, \widehat{\lambda}^{-1}] = e^K, \quad [\widehat{x}^i, \widehat{x}^{\bar{j}}] = -e^K g^{i\bar{j}} \quad (2.27)$$

Then exponentiating these derivative operators, one can construct all of the coherent states by acting with creation operators on a fixed vacuum:

$${}_{(t, \bar{t})}\langle x, \lambda| = {}_{(t, \bar{t})}\langle \Omega | \exp\left[e^{-K} \left(-\lambda^{-1} \widehat{\lambda}^{-1} + x^i g_{i\bar{j}} \widehat{x}^{\bar{j}}\right)\right], \quad (2.28)$$

⁴It should be stressed that, despite the adjective “holomorphic”, Ψ_V does depend on \bar{t} ; the holomorphy here refers to the dependence on the phase space coordinates (x, λ) only; the coordinates (t, \bar{t}) are not part of the phase space we are quantizing, but rather parameterize the possible polarizations.

where $\langle \Omega | = \langle x^i = 0, \lambda^{-1} = 0 |$.

Substituting (2.28) in (2.22), we conclude that the topological wave function in Verlinde’s presentation can be written as an overlap

$$\Psi_V(t, \bar{t}; x, \lambda) = {}_{(t, \bar{t})} \langle \Omega | \exp \left[e^{-K} \left(-\lambda^{-1} \widehat{\lambda}^{-1} + x^i g_{i\bar{j}} \widehat{\bar{x}}^{\bar{j}} \right) \right] | \Psi \rangle. \quad (2.29)$$

To reduce clutter, we will sometimes drop the explicit (t, \bar{t}) dependence of coherent states and creation/annihilation operators from the notation.

We conclude this section with two comments. First, note that the Hermitian metric constructed from ω and \mathcal{J}_t is not positive definite, but rather has signature $(n_v, 1)$ (as reflected by the wrong-sign Gaussian in the integrand of (2.18)). This point, well known in the mathematics literature and also noted *e.g.* in [17], implies that the coherent states we considered above are not normalizable states. Hence it seems that these coherent states must be considered as somewhat formal objects. To perform the quantization rigorously in this indefinite-signature polarization, one should perhaps introduce higher cohomology instead of trying to represent the states as holomorphic functions. Alternatively, one could switch to a polarization for which the metric is positive definite. There is a natural choice available, namely the “Weil” complex structure \mathcal{J}'_t , given by the Hodge \star operator on $\Omega^3(X, \mathbb{R})$, which acts as

$$\mathcal{J}'_t = \begin{cases} +i \text{ on } H^{3,0} \oplus H^{1,2}, \\ -i \text{ on } H^{0,3} \oplus H^{2,1}. \end{cases} \quad (2.30)$$

Formally, one can transform from the \mathcal{J}_t polarization to the \mathcal{J}'_t polarization by Fourier transforming over λ^{-1} (see appendix B). One complication in this approach is that \mathcal{J}'_t does not vary holomorphically with t (unlike \mathcal{J}_t). Throughout this section we will work formally in the indefinite-signature polarization, which is the one where direct contact can be made with the anomaly equations .

Second, let us comment on the occurrence of the line bundle $\mathcal{L}^{-\frac{n_v+1}{2}} \otimes \mathcal{K}_{\mathcal{M}}^{\frac{1}{2}}$ pulled back from \mathcal{M} , where both Ψ_V and the coherent states ${}_{(t, \bar{t})} \langle x, \lambda |$ are valued. This bundle, which we call $\mathcal{K}^{\frac{1}{2}}$, is naturally isomorphic to a square root of $\bigwedge^{n_v+1} (\mathcal{T}^* \otimes \mathcal{L}^* \oplus \mathcal{L}^*)$. In other words, an element of $\mathcal{K}^{\frac{1}{2}}$ is the same as a square-root of a holomorphic top-form on H^3_+ . The wave function Ψ_V is hence naturally “twisted” by the bundle of half-forms. Such a twisting makes no essential difference if one looks only at a fixed t , but it introduces phase factors in the Bogoliubov transformations which arise upon varying t ; indeed, twisting by $\mathcal{K}^{\frac{1}{2}}$ is exactly what is needed to ensure that the wave function has trivial monodromy upon traveling around a contractible loop in \mathcal{M} . This is known as the “metaplectic correction” to the quantization of the phase space (see *e.g.* [18, 19]), and plays an important rôle in making the topological amplitude covariant under electric-magnetic duality.

2.4 Homogeneous coordinates

To simplify the anomaly equations further, it is convenient to replace \mathcal{M} by the space $\widehat{\mathcal{M}}$ of complex structures on X with a chosen holomorphic 3-form Ω . The benefit of so doing is that $\widehat{\mathcal{M}}$ admits a class of particularly nice coordinate systems. To construct them, first note

that $\widehat{\mathcal{M}}$ can be embedded into $H^3(X, \mathbb{C})$ just by mapping a point of $\widehat{\mathcal{M}}$ to its corresponding Ω . This embedding realizes $\widehat{\mathcal{M}}$ as a complex Lagrangian cone inside $H^3(X, \mathbb{C})$.

Suppose we choose a symplectic basis (or “marking”) $\{A^I, B_I\}$, $I = 0, \dots, n_v$, for $H^3(X, \mathbb{R})$. Then define coordinates on $H^3(X, \mathbb{C})$ by

$$X^I = \int_{A^I} \Omega, \quad F_I = \int_{B_I} \Omega, \tag{2.31}$$

and introduce the “characteristic function”

$$F = \frac{1}{2} X^I F_I. \tag{2.32}$$

The cone $\widehat{\mathcal{M}}$ is said to be “generated” in this basis by $F(X)$: this means $\widehat{\mathcal{M}}$ is the subset of $H^3(X, \mathbb{C})$ defined by

$$F_I(X) = \frac{\partial F}{\partial X^I}. \tag{2.33}$$

In particular, a point (X^I, F_I) on $\widehat{\mathcal{M}}$ is determined locally by its X^I , so the X^I give a complex coordinate system on $\widehat{\mathcal{M}}$, which can also be viewed as a homogeneous coordinate system on \mathcal{M} . We also introduce

$$\tau_{IJ}(X) = \partial_I \partial_J F, \tag{2.34}$$

$$C_{IJK}(X) = \partial_I \partial_J \partial_K F. \tag{2.35}$$

In what follows it will be convenient to think of Ψ_V as defined on $\widehat{\mathcal{M}}$; concretely, we consider Ψ_V to depend on the X^I as independent variables, and to be invariant under their overall rescaling.

We expand $\gamma \in H^3(X, \mathbb{R})$ in terms of the basis $\{\partial_I \Omega\}$ instead of the basis $\{\Omega, \delta_i \Omega\}$, writing

$$\gamma = \frac{1}{2} \left(e^{-\frac{1}{4}\pi i} x^I \partial_I \Omega + e^{\frac{1}{4}\pi i} \bar{x}^I \bar{\partial}_I \bar{\Omega} \right). \tag{2.36}$$

where the phase was inserted for later convenience. These coordinates x^I on $H^3(X, \mathbb{R})$ are related to the (x^i, λ^{-1}) of (2.19) by

$$\frac{1}{2} e^{-\frac{1}{4}\pi i} x^I = \lambda^{-1} X^I + x^i D_i X^I, \tag{2.37}$$

where we introduced

$$D_i X^I = \int_{A^I} \delta_i \Omega = (\partial_i + \partial_i K) X^I. \tag{2.38}$$

The x^I are sometimes called “large phase space coordinates”. They give a natural section $\sqrt{\prod_I dx^I}$ of $\mathcal{K}^{\frac{1}{2}}$; we will write Ψ_V with respect to this trivialization.

The symplectic form on $H^3(X, \mathbb{R})$ becomes in these coordinates

$$\omega = \frac{i}{2} [\text{Im } \tau]_{IJ} dx^I \wedge d\bar{x}^J. \tag{2.39}$$

This leads, upon quantization, to the commutators

$$[\widehat{x}^I, \widehat{x}^J] = -2[\text{Im } \tau]^{IJ}. \tag{2.40}$$

Since the change of variable (2.37) is holomorphic (at fixed X), the wave function Ψ_V in the x^I variables is just obtained by direct substitution of the coordinate change into the old wave function, with no Bogoliubov transformation required. In other words, the coherent states ${}_{(t,\bar{t})}\langle x, \lambda |$ which diagonalize $\widehat{\lambda}$ and \widehat{x}^i also diagonalize \widehat{x}^I . Then substituting in (2.26) gives directly

$$\langle x^I | \widehat{x}^I = \langle x^I | x^I, \quad \langle x^I | \widehat{x}^J = 2[\text{Im } \tau]^{JK} \frac{\partial}{\partial x^K} \langle x^I |. \quad (2.41)$$

Similarly, substituting in (2.28) gives

$$\langle x^I | = {}_X \langle \Omega | \exp \left[\frac{1}{2} x^I [\text{Im } \tau]_{IJ} \widehat{x}^J \right]. \quad (2.42)$$

Here ${}_X \langle \Omega |$ is the same vacuum that appeared in (2.28), which is annihilated by \widehat{x}^I .⁵ Thus the overlap formula for Ψ_V , (2.29), may be rewritten as

$$\Psi_V(X^I, \bar{X}^I; x^I) = {}_X \langle \Omega | \exp \left[\frac{1}{2} x^I [\text{Im } \tau]_{IJ} \widehat{x}^J \right] | \Psi \rangle. \quad (2.43)$$

Using

$$\omega^{n_v+1} = \det[\text{Im } \tau] dx^I d\bar{x}^I = |g| e^{-(n_v+1)K} dx^i d\bar{x}^{\bar{i}} d\lambda^{-1} d\bar{\lambda}^{-1} \quad (2.44)$$

and

$$\det[\text{Im } \tau] = |g| e^{-(n_v+1)K}, \quad (2.45)$$

the inner product (2.18) becomes

$$\langle \Psi' | \Psi \rangle = \int dx^I d\bar{x}^I \sqrt{\det[\text{Im } \tau]} e^{-\frac{1}{2} x^I [\text{Im } \tau]_{IJ} \bar{x}^J} \Psi_V'^*(\bar{x}^I) \Psi_V(x^I). \quad (2.46)$$

Finally, the anomaly equations (2.15), (2.16) become [8, 17]

$$\left[\frac{\partial}{\partial \bar{X}^I} + \frac{i}{2} \bar{C}_I^{JK} \frac{\partial^2}{\partial x^J \partial x^K} \right] \Psi_V = 0, \quad (2.47)$$

$$\left[\frac{\partial}{\partial X^I} - \frac{1}{2} \frac{\partial}{\partial X^I} \log \det[\text{Im } \tau] + \frac{i}{2} C_{IJ}^K x^J \frac{\partial}{\partial x^K} + \frac{i}{8} C_{IJK} x^J x^K \right] \Psi_V = 0, \quad (2.48)$$

where the indices are raised and lowered with the metric $[\text{Im } \tau]_{IJ}$ and its inverse $[\text{Im } \tau]^{IJ}$.

2.5 The real polarized topological amplitude

An obvious nuisance in representing $|\Psi\rangle$ in the holomorphic polarizations given by \mathcal{J}_t is the dependence on the basepoint $t \in \mathcal{M}$, which as we have reviewed is responsible for the holomorphic anomaly equations. An alternative (emphasized in [8]) is to consider instead a real polarization, based on the decomposition of $\gamma \in H^3(X, \mathbb{R})$ with respect to a symplectic basis (A^I, B_I) of $H_3(X, \mathbb{R})$, with dual basis (α_I, β^I) :

$$\gamma = p^I \alpha_I + q_I \beta^I. \quad (2.49)$$

⁵We are suppressing the \bar{X} dependence of ${}_X \langle \Omega |$ to lighten the notation.

This gives coordinates (p^I, q_I) on $H^3(X, \mathbb{R})$ which do not require a choice of t ; we have traded that dependence for the dependence on the chosen symplectic basis. The relation between the decomposition (2.49) and the one considered above in (2.36) is

$$p^I = \text{Re}(x^I), \quad q_I = \text{Re}(\tau_{IJ}x^J); \quad (2.50)$$

so upon quantization, p^I and q_I should become Hermitian operators, with

$$\widehat{p}^I = \text{Re}(\widehat{x}^I), \quad \widehat{q}_I = \text{Re}(\tau_{IJ}\widehat{x}^J), \quad (2.51)$$

while the symplectic form $\omega = dq_I \wedge dp^I$ gives

$$[\widehat{p}^I, \widehat{q}_J] = i\delta_J^I. \quad (2.52)$$

So now we can introduce coherent states $|p^I\rangle$ with

$$\widehat{p}^I|p^I\rangle = p^I|p^I\rangle, \quad \widehat{q}_I|p^J\rangle = i\frac{\partial}{\partial p^I}|p^J\rangle, \quad (2.53)$$

normalized to obey the completeness relation

$$\int dp^I |p^I\rangle\langle p^I| = \mathbf{1}. \quad (2.54)$$

If we write $|\Omega\rangle_{\mathbb{R}} = |p^I = 0\rangle$, then the coherent states can be constructed from $|\Omega\rangle_{\mathbb{R}}$ by

$$|p^I\rangle = \exp(-ip^I\widehat{q}_I)|\Omega\rangle_{\mathbb{R}}. \quad (2.55)$$

Using these coherent states one can represent $|\Psi\rangle$ in a real polarization:

$$\Psi_{\mathbb{R}}(p^I) = \langle p^I|\Psi\rangle. \quad (2.56)$$

This $\Psi_{\mathbb{R}}$ is related to $\Psi_V(p^I; X, \bar{X}; x) = \langle x^I|\Psi\rangle$ by a Bogoliubov transformation,

$$\Psi_V(X, \bar{X}; x) = \int dp^I {}_{(X^I, \bar{X}^I)}\langle x^I|p^I\rangle \Psi_{\mathbb{R}}(p^I). \quad (2.57)$$

To write this transformation explicitly we need to determine the intertwining function ${}_{(X^I, \bar{X}^I)}\langle x^I|p^I\rangle$. This is the holomorphically polarized form of the state $|p^I\rangle$, so we call it $\Psi_V(p^I; X, \bar{X}; x)$. Substituting the expressions (2.42), (2.55) for the coherent states, we have

$$\Psi_V(p^I; X, \bar{X}; x) = {}_X\langle\Omega| \exp\left(\frac{1}{2}x^J[\text{Im}\tau]_{JK}\widehat{x}^K\right) \exp(-ip^I\widehat{q}_I)|\Omega\rangle_{\mathbb{R}}. \quad (2.58)$$

Then using $\widehat{x}^I = 2\widehat{p}^I - \widehat{x}^I$, $\widehat{q}_I = \bar{\tau}_{IJ}\widehat{p}^J + i[\text{Im}\tau]_{IJ}\widehat{x}^I$ and commuting the operators \widehat{x}^I , \widehat{p}^J (such that $[\widehat{x}^I, \widehat{p}^J] = -\text{Im}\tau^{IJ}$) to annihilate the vacua at the two ends gives

$$\Psi_V(p^I; X, \bar{X}; x) = ({}_X\langle\Omega|\Omega\rangle_{\mathbb{R}}) \exp\left[-\frac{i}{2}p^I\bar{\tau}_{IJ}p^J + p^I[\text{Im}\tau]_{IJ}x^J - \frac{1}{4}x^I[\text{Im}\tau]_{IJ}x^J\right]. \quad (2.59)$$

The normalization factor ${}_X\langle\Omega|\Omega\rangle_{\mathbb{R}}$ can be determined (up to an unimportant overall constant) by requiring that $\Psi_V(p^I; X, \bar{X}; x)$ obeys the equations (2.47), (2.48): it turns out to be $\sqrt{\det[\text{Im } \tau]}$, so altogether⁶

$$\Psi_V(p^I; X, \bar{X}; x) = \sqrt{\det[\text{Im } \tau]} \exp \left[-\frac{i}{2} p^I \bar{\tau}_{IJ} p^J + p^I [\text{Im } \tau]_{IJ} x^J - \frac{1}{4} x^I [\text{Im } \tau]_{IJ} x^J \right]. \tag{2.60}$$

In appendix A, we discuss how the real polarization relates to the ‘‘holomorphic ambiguities’’ which appear in [5, section 7].

We close with one comment. Above we have emphasized that the real polarization involves the choice of a symplectic basis of 3-cycles (symplectic marking). As usual in quantum mechanics, the wave functions $\Psi_{\mathbb{R}}, \Psi_{\mathbb{R}'}$ in two different symplectic bases differ by a Bogoliubov transformation: *e.g.* under $B_I \rightarrow B_I + m_{IJ} A^J$, $\Psi_{\mathbb{R}}$ transforms by

$$\Psi_{\mathbb{R}'}(p^I) = e^{ip^I m_{IJ} p^J} \Psi_{\mathbb{R}}(p^I), \tag{2.61}$$

and under an exchange $A_I \leftrightarrow B^I$,

$$\Psi_{\mathbb{R}'}(q_I) = \int dp^I e^{ip^I q_I} \Psi_{\mathbb{R}}(p^I). \tag{2.62}$$

There is a small subtlety here: it turns out that in order to fix some sign ambiguities in the Bogoliubov transformations one needs to consider the wave function as a half-density. So it is $\Psi_{\mathbb{R}}(p^I) \sqrt{|\prod dp^I|}$ that is actually well defined [18]. This is another aspect of the metaplectic correction which we mentioned in section 2.3.

2.6 The (anti)holomorphic topological amplitude

So far we have mainly reviewed forms of the topological string partition function already found in the literature. In the next two subsections we introduce two new variants.

We begin with the holomorphically polarized topological amplitude (2.43),

$$\Psi_V(X^I, \bar{X}^I; x^I) = {}_X\langle\Omega| \exp \left[\frac{1}{2} x^I [\text{Im } \tau]_{IJ} \widehat{x}^J \right] |\Psi\rangle. \tag{2.63}$$

As we have discussed, this Ψ_V is a section of the holomorphic line bundle $\mathcal{K}^{\frac{1}{2}}$ over the complex manifold $\widetilde{\mathcal{M}}$, the total space of \mathcal{H}_+^3 over \mathcal{M} . It obeys holomorphic anomaly equations which are close to, but not identical to, those satisfied by the complex conjugate of a theta function. The obvious difference is that a theta function $\theta(t, z)$ is purely holomorphic both in t and in z . So we want to make some transformation so that Ψ_V will become purely antiholomorphic.

To do so, we need to overcome three obstacles, two of which are apparent even without looking at the detailed form of the anomaly equations. First, $\mathcal{K}^{\frac{1}{2}}$ is a holomorphic line bundle over $\widetilde{\mathcal{M}}$, so there is no well defined notion of an antiholomorphic section. Second, Ψ_V depends holomorphically rather than antiholomorphically on the fiber coordinates x^I .

⁶This function was first obtained in [17], by factorizing the semiclassical partition function of the NS5-brane wrapped on X .

Both of these problems can be solved using the Hermitian metric in \mathcal{H}_+^3 (which in our coordinates is $[\text{Im } \tau]_{IJ}$.) First, using this metric one can identify $\mathcal{K}^{\frac{1}{2}} \simeq \widehat{\mathcal{K}}^{-\frac{1}{2}}$, so that we will have a section of an antiholomorphic line bundle instead of a holomorphic one. In our coordinates and local trivialization this just means dividing by $\sqrt{\det[\text{Im } \tau]}$. Second, one can also identify the total space of \mathcal{H}_+^3 with that of $(\widehat{\mathcal{H}}_+^3)^* = (\mathcal{H}_-^3)^*$, thus reversing the complex structure on the fibers, so that Ψ_V will be antiholomorphic on them. This amounts to the change of coordinate $x^I = [\text{Im } \tau]^{IJ} \widehat{y}_J$.

Making these transformations in (2.63) leads to the function

$${}_X \langle \Omega_{\text{ah}} | \exp \left[\frac{1}{2} \widehat{y}_J \widehat{x}^J \right] | \Psi \rangle. \tag{2.64}$$

Here we have rescaled $\langle \Omega | = \sqrt{\det[\text{Im } \tau]} \langle \Omega_{\text{ah}} |$. Still, (2.64) cannot be antiholomorphic in X : the trouble now is with the annihilation operator \widehat{x}^J , which we recall depends implicitly on X . But since \widehat{x}^J annihilates $\langle \Omega_{\text{ah}} |$ one may replace \widehat{x}^J by $\widehat{x}^J + \widehat{x}^J = 2\widehat{x}^J$, at the expense of introducing an extra phase from the Baker-Campbell-Hausdorff formula in commuting \widehat{x}^J past \widehat{x}^J :

$${}_X \langle \Omega_{\text{ah}} | \exp \left[\widehat{y}_J \widehat{p}^J \right] | \Psi \rangle = \exp \left(\frac{1}{4} x^I [\text{Im } \tau]_{IJ} x^J \right) {}_X \langle \Omega_{\text{ah}} | \exp \left[\frac{1}{2} \widehat{y}_J \widehat{x}^J \right] | \Psi \rangle. \tag{2.65}$$

The left side of (2.65) is our candidate for the antiholomorphic version of the topological amplitude:

$$\Psi_{\text{ah}}(X^I, \bar{X}^I; \bar{y}_I) = {}_X \langle \Omega_{\text{ah}} | \exp \left[\widehat{y}_I \widehat{p}^I \right] | \Psi \rangle. \tag{2.66}$$

In terms of the original Ψ_V , we can summarize our transformation as

$$\Psi_{\text{ah}}(X^I, \bar{X}^I; \bar{y}_I) = \frac{\exp \left(\frac{1}{4} x^I [\text{Im } \tau]_{IJ} x^J \right)}{\sqrt{\det[\text{Im } \tau]}} \Psi_V(X^I, \bar{X}^I; x^J). \tag{2.67}$$

Now we want to see what sort of anomaly equations Ψ_{ah} obeys. The simplest way to work this out is to apply the transformation (2.67) to the explicit solutions (2.60) giving the intertwiner to the real polarization: they become

$$\Psi_{\text{ah}}(p^I; X, \bar{X}; \bar{y}_I) = \exp \left[-\frac{i}{2} p^I \bar{\tau}_{IJ} p^J + p^I \bar{y}_I \right]. \tag{2.68}$$

To avoid the nuisance of working with antiholomorphic objects, from now on we use the holomorphic $\Psi_{\text{hol}} = \Psi_{\text{ah}}^*$, a section of $\mathcal{K}^{-\frac{1}{2}}$. Then in terms of Ψ_{hol} , the anomaly equations take the simplified form⁷

$$\left[\frac{\partial}{\partial X^I} - \frac{i}{2} C_{IJK} \frac{\partial^2}{\partial y_J \partial y_K} \right] \Psi_{\text{hol}} = 0, \tag{2.69}$$

$$\frac{\partial}{\partial X^I} \Psi_{\text{hol}} = 0. \tag{2.70}$$

⁷Note that (2.69), (2.70) could be summarized as $\delta \Psi_{\text{hol}} = \frac{i}{2} \delta \tau_{JK} \partial_{y_J} \partial_{y_K} \Psi_{\text{hol}}$, which is formally identical to the heat equation giving the variation of a holomorphic Siegel theta function $\Theta(\tau, y)$ with τ . Indeed Ψ_{hol} behaves very much like a Siegel theta function, except that τ is not allowed to vary arbitrarily, and worse yet $\text{Im } \tau$ is not positive definite, so the usual construction of holomorphic Siegel theta functions would lead to a divergent series.

The inner product (2.46) of wave functions becomes

$$\langle \Psi' | \Psi \rangle = \int dy_I d\bar{y}_I \sqrt{\det[\text{Im } \tau]} e^{-\frac{1}{2}(y_I + \bar{y}_I)[\text{Im } \tau^{-1}]^{IJ}(y_J + \bar{y}_J)} \Psi'_{\text{hol}}(X, \bar{X}; y_I) \Psi_{\text{hol}}(X, \bar{X}; y_I). \tag{2.71}$$

2.7 The generalized heat equation

For comparison to the results we will find in section 3 we will make one further transformation of the partition function. This transformation can be made only after choosing a privileged element A^0 in $H_3(X, \mathbb{R})$, with corresponding period $\int_{A^0} \Omega = X^0$. Having done so, one may define coordinates on \mathcal{M} by

$$t^i = \frac{X^i}{X^0}. \tag{2.72}$$

Now we make a small further adjustment: we divide the coordinates y_I into $\{y_0, y_i\}$, and then replace y_0 by

$$w = y_0 + t^i y_i. \tag{2.73}$$

(This shift will find a natural interpretation in section 3.5.) Considering Ψ_{hol} as a function of (t^i, y_i, w) , (2.69), (2.70) then become

$$\left[\frac{\partial}{\partial t^i} - \frac{i}{2} C_{ijk} \frac{\partial^2}{\partial y_j \partial y_k} + y_i \frac{\partial}{\partial w} \right] \Psi_{\text{hol}} = 0, \tag{2.74}$$

$$\frac{\partial}{\partial \bar{t}^i} \Psi_{\text{hol}} = 0. \tag{2.75}$$

This is not too hard to check using the relation $X^I C_{IJK} = 0$, which follows from the definition (2.35) and the homogeneity of F , and which in these coordinates implies $p^J p^K C_{iJK} = (p^j - p^0 t^j)(p^k - p^0 t^k) C_{ijk}$.

Altogether then, we have found that the complex conjugate of the BCOV topological string amplitude can be transformed into a strictly holomorphic function Ψ_{hol} , obeying a heat equation analogous to that of the Jacobi theta series. To be precise, we gave two closely related versions of this heat equation, namely (2.69) and (2.74).

The existence of such a heat equation is not unexpected: indeed, it was the similarity between the anomaly equation of [6] and the heat equation that led Witten to propose the wave function interpretation of the topological string partition function. This interpretation was natural particularly in light of the earlier work [20], in which the heat equation for (abelian and non-abelian) theta functions was identified as a “background dependence” in the quantization of the moduli space of flat connections on a Riemann surface, coming from the need to choose a complex structure. However, the analogy has always been slightly puzzling: if the topological string partition function arose in this way one would expect it to be purely holomorphic, as was the case for the theta functions of [20]. We view the transformations relating Ψ_{BCOV}^* to Ψ_{hol} , given above, as the resolution of this small puzzle: in summary, Ψ_{BCOV}^* can be made holomorphic by choosing the proper complex structure on its domain, and by choosing a different convention for the genus zero 0,1,2-point functions than was chosen in [6].

In section 3, we will give a representation-theoretic construction of the anomaly equations for Ψ_{hol} , exhibiting its wave function nature directly, in cases where \mathcal{M} is a symmetric space.

2.8 Modularity

So far we have considered the topological string partition function on the Teichmüller space \mathcal{M} and its extension to $\widetilde{\mathcal{M}}$. In the string theory, though, the relevant space is generally not \mathcal{M} but some quotient $\Gamma \backslash \mathcal{M}$, where Γ is a “modular group” of large biholomorphisms of X . So we should ask how the topological string partition function transforms under Γ .

For the original partition function Ψ_{BCOV} the answer is simple. Since Ψ_{BCOV} is defined directly in terms of the B model topological string theory, which appears to be Γ -invariant, Ψ_{BCOV} should also be Γ -invariant. However, we must be careful about what this means, since Ψ_{BCOV} is a section of $\mathcal{L}^{1-\frac{\chi}{24}}$. So to be precise, the action of Γ on X induces holomorphic actions on \mathcal{M} , \mathcal{T} and \mathcal{L} , and we expect that Ψ_{BCOV} is invariant under the combined action. For the modified partition function Ψ_V the story is similar, except that Ψ_V is a section of $\mathcal{K}^{\frac{1}{2}}$ instead of $\mathcal{L}^{1-\frac{\chi}{24}}$.

If we choose some specific coordinates on $H^3(X, \mathbb{C})$ and trivialization of $\mathcal{K}^{\frac{1}{2}}$, then we can be more concrete. For example, suppose we choose the coordinates x^I and trivialization of $\mathcal{K}^{\frac{1}{2}}$ by the half-form $\sqrt{\prod_I dx^I}$, as in section 2.4. An element $\gamma \in \Gamma$ acts on $H^3(X, \mathbb{C})$ by some element in $Sp(2n_v + 2, \mathbb{Z})$, which we also write as γ : in a basis where the symplectic form is the block-diagonal $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, write

$$\gamma = \begin{pmatrix} A^I_J & B_{IJ} \\ C^{IJ} & D^I_J \end{pmatrix}. \tag{2.76}$$

Based on the above we expect that $\Psi_V(t, x) \sqrt{\prod_I dx^I}$ should be invariant under the action of γ . This leads to the transformation law

$$(\det[C\tau + D])^{\frac{1}{2}} \Psi_V(\gamma t, (C\tau + D)x) = \Psi_V(t, x). \tag{2.77}$$

The point γt can be characterized by the fact that the period matrix τ_{IJ} transforms as $\gamma\tau = (A\tau + B)(C\tau + D)^{-1}$.

Now what about Ψ_{hol} ? The transformation (2.67) relating Ψ_V to Ψ_{hol}^* introduces extra factors which depend explicitly on the choice of a symplectic basis. With these factors included, the transformation law (2.77) is modified to

$$(\det[C\tau + D])^{-\frac{1}{2}} \Psi_{\text{hol}}(\gamma t, (C\tau + D)^{-1}y) = \exp\left(-\frac{i}{2}y_I [(C\tau + D)^{-1}]^I_J C^{JK} y_K\right) \Psi_{\text{hol}}(t, y), \tag{2.78}$$

just as for the Riemann theta series [21].

This modular property in principle gives strong constraints on Ψ_{hol} . One concrete way to extract these constraints would be to make a perturbative expansion of $\log \Psi_{\text{hol}}$. At each genus we would then expect to find a holomorphic section of a holomorphic line bundle over $\Gamma \backslash \mathcal{M}$. If $\Gamma \backslash \mathcal{M}$ can be compactified while keeping control of the behavior near

the boundary points, then such a line bundle has a finite-dimensional space of holomorphic sections; in other words, Ψ_{hol} is determined up to a finite-dimensional ambiguity at each genus. This seems to be consistent with the results of [15, 16], who have recently studied the constraints modular invariance and the anomaly equation impose on topological string perturbation theory.

3. The holomorphic anomaly for symmetric tube domains

Now we consider a special class of $\mathcal{N} = 2$, $d = 4$ supergravity theories coupled to vector multiplets. These theories are constructed from Euclidean Jordan algebras J with invariant cubic norms, and can be obtained by dimensional reduction from corresponding theories in five dimensions, which were constructed in [22–24]. In the five-dimensional description, the vector fields are in 1-1 correspondence with basis elements of J . After dimensional reduction this correspondence is extended: both electric and magnetic field strengths get related to basis elements of the “Freudenthal triple system” defined over J [25–29].

Given J we review the construction of this Freudenthal triple system as well as a special Kähler manifold \mathcal{M} , which is the vector multiplet moduli space of the corresponding $d = 4$ supergravity theory, and a global symmetry group G which acts transitively on \mathcal{M} by isometries. \mathcal{M} is equipped with a privileged set of special coordinates, in which the prepotential is directly related to the cubic norm of J .

Most of these supergravity theories are not known to arise as low energy limits of Type II string theory compactified on a Calabi-Yau threefold. One notable exception comes from Type II string theory compactified on a quotient of $K3 \times T^2$, considered in [30] and recently reinvestigated in [31, 32]. The vector multiplet moduli space of this theory has a classical $SO(10, 2) \times SU(1, 1)$ symmetry acting transitively. *A priori* this symmetry could be broken by genus zero worldsheet instantons, but it turns out that these instantons are absent, so the exact special geometry of the moduli space is $SO(10, 2) \times SU(1, 1)$ symmetric. Indeed, this model gives a string theory realization of one of the generic Jordan family of $\mathcal{N} = 2$ Maxwell-Einstein supergravity theories, with $n = 11$ vector multiplets (the first row of Figure 1 below), coupled to 12 hypermultiplets.

It would be interesting to have more examples of magical supergravity theories which can be embedded into string theory. In fact, several candidate theories can be found in [33], among them a model which at low energy describes the coupling of 15 vector multiplets to $\mathcal{N} = 2$ supergravity. It is easy to show that, at least before considering worldsheet instantons, this is the “magical” supergravity theory with moduli space $SO^*(12)/(SU(6) \times U(1))$ (the fifth row in Figure 1). It is not known whether the $SO^*(12)$ symmetry of this moduli space survives into the quantum theory.

In any case, the holomorphic anomaly equation can be constructed from an $\mathcal{N} = 2$, $d = 4$ supergravity theory without reference to its embedding into string theory. From now on we adopt this point of view.

3.1 The phase space

We begin with a Euclidean Jordan algebra J , of real dimension n_v .⁸ We assume that J admits a cubic norm N , which may be written concretely in linear coordinates

$$N(p) = \frac{1}{6} C_{ijk} p^i p^j p^k. \tag{3.1}$$

From J one can construct an associated “Freudenthal triple system” (see *e.g.* [35–41]); this is a real vector space V which decomposes as

$$V = \mathbb{R} \oplus J \oplus J \oplus \mathbb{R}. \tag{3.2}$$

We choose linear coordinates p^0, p^i, q_i, q_0 respectively on the four summands (using the Jordan algebra trace to pair the two copies of J). We also introduce the index $I = (0, i)$. Then V is equipped with a natural symplectic form ω ,

$$\omega = dp^0 \wedge dq_0 + dp^i \wedge dq_i = dp^I \wedge dq_I, \tag{3.3}$$

and a quartic form constructed from the norm on J ,

$$I_4 = 4p^0 N(q_i) - 4q_0 N(p^i) + 4q_{\sharp}^i p_i^{\sharp} - (p^0 q_0 + p^i q_i)^2. \tag{3.4}$$

where $p_i^{\sharp} := \frac{1}{2} C_{ijk} p^j p^k$, $q_{\sharp}^i := \frac{1}{2} C^{ijk} q_j q_k$. We define G as the group of linear transformations of V which preserve ω and I_4 ; this G shows up as a global symmetry of the supergravity theory, and in particular it acts by isometries on the moduli space \mathcal{M} as we will see below.

There is a particularly important generator $D \in \mathfrak{g}$ which recovers the real decomposition (3.2), acting with eigenvalues $-3, -1, +1, +3$ respectively on the four summands. It also induces a decomposition of \mathfrak{g} into real subspaces

$$\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_+ \tag{3.5}$$

where $\mathfrak{g}_-, \mathfrak{g}_0, \mathfrak{g}_+$ have eigenvalues $-2, 0, 2$ respectively. The three subspaces can be conveniently described in terms of J . Namely, \mathfrak{g}_0 is the “structure algebra” of J , consisting of infinitesimal linear transformations of J preserving N up to overall rescaling; these act on V in the obvious way. \mathfrak{g}_- and \mathfrak{g}_+ are isomorphic as vector spaces to J ; we write the generators as S_i and T^i respectively, and then define $R_i^j = [S_i, T^j]$. The trace of R gives the abovementioned D , as $D = \frac{3}{n_v} R_i^i$; we also define the traceless part $D_j^i = R_j^i - \frac{1}{3} D$.

3.2 The Teichmüller space

Next we need to describe the Teichmüller space \mathcal{M} . As we mentioned above, this space has been studied extensively in the physics literature [22, 23, 42–46]; for a more mathematical review and pointers to the mathematical literature see *e.g.* the book [47]. Here we present a construction of \mathcal{M} described in [48], which is tailored to our needs: it mimics the procedure

⁸A review of the relevant aspects of Jordan algebras can be found in [23, 34]; in what follows we will not need to know much about their algebraic structure.

of section 2.4, where the Teichmüller space of a Calabi-Yau threefold X was realized as the projectivization of a Lagrangian cone $\widehat{\mathcal{M}}$ inside the complexified phase space $H^3(X, \mathbb{C})$.

In our current setup V is playing the role of $H^3(X, \mathbb{R})$. So we should begin by complexifying the phase space V to $V_{\mathbb{C}}$. Then the action of G on V extends holomorphically to an action of a complexified group $G_{\mathbb{C}}$ on $V_{\mathbb{C}}$. Inside $V_{\mathbb{C}}$, we consider the orbit \mathcal{O} of the “lowest weight” vector $(p^0, p^i, q_i, q_0) = (1, 0, 0, 0)$. From the fact that any element of $\mathfrak{g}_{\mathbb{C}}$ only changes the D -grading by 0 or ± 2 , it follows that \mathcal{O} is a Lagrangian cone; the projectivization of \mathcal{O} is a compact manifold which we call \mathcal{M}_D . This is almost, but not quite, the desired \mathcal{M} : indeed, by construction \mathcal{M}_D is a single orbit of $G_{\mathbb{C}}$, but under the action of the real group $G \subset G_{\mathbb{C}}$ it has a further decomposition into orbits. The desired \mathcal{M} is a “big” orbit (open in \mathcal{M}_D) for which the stabilizer of any point is a maximal compact subgroup of G . By construction, then, $\mathcal{M} \simeq G/K$ with K a maximal compact subgroup of G . Similarly we can write \mathcal{M}_D as $G_{\mathbb{C}}/P$ with P the stabilizer of $(1, 0, 0, 0)$. It will be useful below to remember that \mathcal{M} is embedded into \mathcal{M}_D .⁹ In particular, this embedding immediately shows that \mathcal{M} has a complex structure on which G acts by holomorphic automorphisms. Moreover, it gives a natural notion of analytic continuation to the boundary of \mathcal{M} .

This construction of \mathcal{M} inside $\mathbb{P}(V_{\mathbb{C}})$ gives a nice class of homogeneous coordinate systems on \mathcal{M} , coming from symplectic markings of V , just as we discussed in section 2.4 for symplectic markings of $H^3(X, \mathbb{R})$. Actually, in this case the situation is even better: the decomposition (3.2) of V gives a natural symplectic marking of V , which turns out to give an especially convenient projective coordinate system X^I . The prepotential F (defined as the function which generates the Lagrangian cone in these coordinates, as in section 2.4) then turns out to be purely cubic,

$$F = \frac{N(X^i)}{X^0}. \tag{3.6}$$

The decomposition (3.2) also gives a natural 1-dimensional subspace (hence a natural choice of the 0-th direction), which one can use to define honest coordinates on \mathcal{M} ,

$$t^i = \frac{X^i}{X^0}. \tag{3.7}$$

For example, taking the Jordan algebra $J = \mathbb{R}$, the above construction gives \mathcal{M} as the upper half-plane with its standard coordinate $t = x + iy$. More generally, the coordinates $t^i = x^i + iy^i$ realize \mathcal{M} as a “generalized (Köcher) upper half-plane” inside $J_{\mathbb{C}}$, of the

⁹To make this embedding $\mathcal{M} \subset \mathcal{M}_D$ seem more familiar, we recall a closely related example. Consider $G_{\mathbb{C}} = SL(2, \mathbb{C})$ acting on $V_{\mathbb{C}} = \mathbb{C}^2$. Then the orbit \mathcal{O} of the vector $(0, 1)$ is \mathbb{C}^2 minus the origin. Projectivizing gives the Riemann sphere $\mathbb{C}\mathbb{P}^1$ as \mathcal{M}_D . Then letting $SL(2, \mathbb{C})$ act on the Riemann sphere by linear fractional transformations, the stabilizer P of the south pole consists of lower-triangular matrices; so \mathcal{M}_D is $SL(2, \mathbb{C})/P$. On the other hand, if one considers just the action of $SL(2, \mathbb{R})$, then \mathcal{M}_D breaks up into orbits, namely the two hemispheres and the equator. On either the top or bottom hemisphere, the stabilizer in $SL(2, \mathbb{R})$ of a point is $U(1)$; we call the top hemisphere \mathcal{M} . So the standard realization of the upper half plane \mathcal{M} as $SL(2, \mathbb{R})/U(1)$ is naturally embedded inside the realization of the Riemann sphere \mathcal{M}_D as $SL(2, \mathbb{C})/P$.

J	G	K	V	n_v
$\mathbb{R} \oplus \Gamma_{n-1,1}$	$SU(1,1) \times SO(2,n)$	$U(1) \times U(1) \times SO(n)$	$(\mathbf{2}, \mathbf{n} + \mathbf{2})$	$n + 1$
\mathbb{R}	$SL(2, \mathbb{R})$	$U(1)$	$\mathbf{4}$	1
$Herm_3(\mathbb{R})$	$Sp(6, \mathbb{R})$	$U(1) \times SU(3)$	$\mathbf{14}$	6
$Herm_3(\mathbb{C})$	$SU(3,3)$	$U(1) \times SU(3) \times SU(3)$	$\mathbf{20}$	9
$Herm_3(\mathbb{H})$	$SO^*(12)$	$U(1) \times SU(6)$	$\mathbf{32}$	15
$Herm_3(\mathbb{O})$	$E_{7(-25)}$	$U(1) \times E_6$	$\mathbf{56}$	27

Table 1: Euclidean Jordan algebras of degree 3 and their corresponding G , K , V and n_v . The groups G , K are given only up to a finite cover or finite quotient. $Herm_3$ denotes 3×3 Hermitian matrices; $\Gamma_{n-1,1}$ is a Jordan algebra of degree 2 defined by a quadratic form of Lorentzian signature [23].

form [23]

$$\mathcal{M} = \{t = x + iy : x \in J, y \in J_+\} \subset J_{\mathbb{C}}, \tag{3.8}$$

where J_+ is the cone of “positive definite” elements in J , with boundary contained in the “cubic light-cone” $\{y : N(y) = 0\}$. \mathcal{M} also has a natural Kähler metric inherited from $V_{\mathbb{C}}$, with the simple Kähler potential

$$K(t^i, \bar{t}^{\bar{i}}) = -\log N(t^i - \bar{t}^{\bar{i}}). \tag{3.9}$$

In these coordinates, the action of G on \mathcal{M} can be described simply, *e.g.* because it comes from the linear action on $V_{\mathbb{C}}$: one finds that the generators T_i, R_j^i, S^i act respectively by translations, Lorentz rotations/dilatations and “special conformal” transformations [49],

$$T_i \mapsto \frac{\partial}{\partial t^i}, \quad R_j^i \mapsto -\Lambda_{jl}^{ik} t^l \frac{\partial}{\partial t^k}, \quad S^i \mapsto \frac{1}{2} \Lambda_{kl}^{ij} t^k t^l \frac{\partial}{\partial t^j}, \tag{3.10}$$

where $\Lambda_{kl}^{ij} = \delta_k^i \delta_l^j + \delta_k^j \delta_l^i - C_{klm} C^{ijm}$.¹⁰ Here we introduced C^{ijk} , where the indices are raised using the metric on \mathcal{M} ; this actually gives a constant tensor, numerically equal to C_{ijk} up to an overall constant, which can be shown using the fact that both C^{ijk} and C_{ijk} descend from invariant tensors in the five-dimensional supergravity theory [24].

Note that the action of G preserves the condition that all t^i are real. So G can be thought of as a “generalized conformal group” $\text{Conf}(J)$ which acts on the Jordan algebra J , as well as on $\mathcal{M} \subset J_{\mathbb{C}}$ [49]. (This generalizes the fact that $SL(2, \mathbb{R})$ acts on \mathbb{R} as well as on the upper half-plane.) The action of G on J preserves separations along the cubic light-cone, i.e. $N(x - x') = 0 \implies N(gx - gx') = 0$.

Now we have finished constructing the phase space V , the Teichmüller space \mathcal{M} , and the group G , which we have seen acts naturally both on V (by linear maps) and on \mathcal{M} (by holomorphic automorphisms). In Figure 1 we tabulate the Euclidean Jordan algebras of degree 3 and their corresponding G , K , V and n_v .

¹⁰Our C differs from that of GST [23] by a factor $2/\sqrt{3}$, $C_{ijk}^{\text{here}} = \frac{2}{\sqrt{3}} C_{ijk}^{\text{GST}}$, to agree with the conventions used in the special geometry literature.

3.3 Quantization of phase space: the Heisenberg group

Following the discussion of section 2, we now want to understand the topological string partition function in these models as a state $|\Psi\rangle$ in the quantization of V . This quantization as usual gives a Hilbert space \mathcal{H}_\hbar acted on by operators corresponding to the coordinates $\{p^I, q_I\} \in V^*$. These operators together with a single central generator Z generate a Heisenberg group H , with the standard commutation relations¹¹

$$[i\widehat{p}^I, i\widehat{q}_J] = -\delta_J^I Z, \tag{3.11}$$

where Z acts on \mathcal{H}_\hbar by a pure imaginary scalar $i\hbar$. In what follows we will always set $\hbar = 1$; this choice is consistent with section 2 where we used the commutation relations $[\widehat{p}^I, \widehat{q}_J] = i\delta_J^I$.

To construct a “topological string partition function” obeying the holomorphic anomaly equations from the abstract state $|\Psi\rangle$, we want to introduce a basis of coherent states for each point $t \in \mathcal{M}$. Hence we need to define a complex structure \mathcal{J}_t on V . We define it as in section 2.3, in terms of a Hodge decomposition. Recall that D gave a real decomposition of V , shown in (3.2). More generally, its conjugate $D_t = (\text{Ad}(\exp t^i T_i))D$ gives a conjugate decomposition

$$V_{\mathbb{C}} = V_3 \oplus V_1 \oplus V_{-1} \oplus V_{-3} \tag{3.12}$$

where for complex t the eigenspaces are not invariant under conjugation. This is our desired Hodge decomposition at the point $t \in \mathcal{M}_D$, from which \mathcal{J}_t can be constructed as in section 2.3. This Hodge decomposition has been discussed previously in [50].

3.4 The Schrödinger-Weil representation

So far we have discussed the action of the Heisenberg group H on the Hilbert space \mathcal{H}_\hbar . In the present case there is another natural group around, namely G , which acts on V by symplectic transformations. The symplectic transformations induce an action on \mathcal{H}_\hbar : $g \in G$ gives a Bogoliubov transformation $B(g)$, satisfying the condition¹²

$$B(g)\widehat{f}B(g)^{-1} = \widehat{g}f. \tag{3.13}$$

The requirement (3.13), together with unitarity, is enough to determine $B(g)$ up to a phase ambiguity. In particular, it implies that $B(g)B(g') = e^{i\theta(g,g')}B(gg')$. One can then try to choose the ambiguous phases in such a way that $B(g)B(g') = B(gg')$. This is not possible in general, but one can always manage it by passing to a “metaplectic” covering group of G ; in what follows we will implicitly replace G by this cover when necessary.

¹¹We write the commutation relations for $i\widehat{p}$ and $i\widehat{q}$ rather than \widehat{p} and \widehat{q} because we want to deal with the skew-Hermitian generators of the real group H .

¹²The existence of some $B(g)$ satisfying (3.13) can be abstractly proven by the following standard argument: for a fixed g , we consider two different actions of H on \mathcal{H}_\hbar , one by $if \mapsto i\widehat{f}$, the other by $if \mapsto i\widehat{g}f$. Because g preserves the symplectic structure, these are both representations of H , with the same central character $Z \mapsto i\hbar$. Hence the Stone-von Neumann Theorem guarantees that they are equivalent, so the desired $B(g)$ exists.

	-3	-2	-1	0	1	2	3
2	Z						
1	$i\widehat{q}_0$	$i\widehat{q}_i$		$i\widehat{p}^i$		$i\widehat{p}^0$	
0	S^i		R_j^i, D		T_i		

Table 2: A list of the generators of the Lie algebra $\widetilde{\mathfrak{g}}$, with two gradings, by D (horizontal) and by Δ (vertical).

So we have actions of G and H on the Hilbert space \mathcal{H}_\hbar , with the commutation between the two determined by (3.13); hence we can think of \mathcal{H}_\hbar as a representation of an extended group $\widetilde{G} = G \ltimes H$. Following [10] we call it the ‘‘Schrödinger-Weil representation’’. At least formally, it has a natural extension to a representation of a complexified group $\widetilde{G}_\mathbb{C} = G_\mathbb{C} \ltimes H_\mathbb{C}$, acting holomorphically on \mathcal{H}_\hbar ; in the construction of the topological amplitude below, we work formally with the complexified generators.

Let us describe \widetilde{G} in a bit more detail. The Lie algebra $\widetilde{\mathfrak{g}}$ has a decomposition which naturally extends (3.5): in addition to the grading by the dilatation generator D which runs from -3 to 3 , there is a second grading Δ (not implemented by any element of $\widetilde{\mathfrak{g}}$, but see section 3.8) under which \mathfrak{g} has grade 0, V^* grade 1, and Z grade 2. We represent the two gradings in Figure 2. Moreover, we can realize the Schrödinger-Weil representation \mathcal{H}_\hbar (with $\hbar = 1$) concretely in the real polarization given by our chosen symplectic marking: then \mathcal{H}_\hbar just consists of L^2 functions of the $n_v + 1$ real variables p^I , and the generators of $\widetilde{\mathfrak{g}}$ act by

$$Z \mapsto i, \tag{3.14}$$

$$i\widehat{q}_0 \mapsto \frac{\partial}{\partial p^0}, \quad i\widehat{q}_i \mapsto \frac{\partial}{\partial p^i}, \quad i\widehat{p}^i \mapsto ip^i, \quad i\widehat{p}^0 \mapsto ip^0, \tag{3.15}$$

$$S^i \mapsto -\frac{i}{2} C^{ijk} \frac{\partial^2}{\partial p^j \partial p^k} + p^i \frac{\partial}{\partial p^0}, \quad T_i \mapsto +\frac{i}{2} C_{ijk} p^j p^k - p^0 \frac{\partial}{\partial p^i}, \tag{3.16}$$

$$R_i^j \mapsto \delta_i^j p^0 \frac{\partial}{\partial p^0} - p^j \frac{\partial}{\partial p^i} + \frac{1}{2} C_{ikl} C^{jnl} \left(p^k \frac{\partial}{\partial p^n} + \frac{\partial}{\partial p^n} p^k \right). \tag{3.17}$$

Now we recall that C , constructed from the norm of the Jordan algebra J in (3.1), satisfies a crucial ‘‘adjoint identity’’ [23] which implies

$$C_{ikl} C^{inl} = \left(\frac{n_v}{3} + 1 \right) \delta_k^n. \tag{3.18}$$

Then defining $D = \frac{3}{n_v} R_i^i$ and using (3.17) we find

$$D \mapsto 3p^0 \frac{\partial}{\partial p^0} + p^i \frac{\partial}{\partial p^i} + \frac{1}{2} (n_v + 3). \tag{3.19}$$

For later use, we note a few facts which can be easily checked using this explicit realization.

First, we introduce a state $|\Omega\rangle_0$ defined by the constant function $f(p) = 1$.¹³ This state is annihilated by the generators \widehat{q}^I ; these generators are not the annihilation operators of

¹³This $|\Omega\rangle_0$ is not normalizable, so it does not strictly speaking belong to \mathcal{H}_\hbar .

any holomorphic polarization, but it turns out that they arise as the $t = 0$ limit of the annihilation operators of \mathcal{J}_t , so we can sensibly refer to $|\Omega\rangle_0$ as a “vacuum at $t = 0$.” Now we note that $|\Omega\rangle_0$ is also annihilated by S^i and D_j^i , and has eigenvalue $\frac{1}{2}(n_v + 3)$ under D .

Second, we note that the operator equation

$$ZT_i + \frac{1}{2}C_{ijk}\widehat{p}^j\widehat{p}^k - \widehat{p}^0\widehat{q}_i = 0 \tag{3.20}$$

holds in the Schrödinger-Weil representation. In the next section we will identify (3.20) with the holomorphic anomaly equation.

3.5 The anomaly equations

In this section we will argue that Ψ_{hol} , when considered as a function of both t and y , has a natural interpretation as a matrix element of the Schrödinger-Weil representation, and that if one adopts this point of view from the outset, the anomaly equations arise naturally from the operator equations (3.20).

First recall from (2.66) that for fixed $t \in \mathcal{M}$, $\Psi_{\text{hol}}(t, y)$ can be understood as a matrix element of H acting on the Hilbert space \mathcal{H}_h , between $|\Psi\rangle$ and a vacuum $|\Omega\rangle_t$ in the \mathcal{J}_t polarization:

$$\Psi_{\text{hol}}(t, y) = \langle \Psi | \exp(y_I \widehat{p}^I) | \Omega \rangle_t. \tag{3.21}$$

On the other hand, as we have just seen, the Bogoliubov transformations which relate the vacua in various polarizations can be implemented by the group $G_{\mathbb{C}}$ acting in the Schrödinger-Weil representation. In particular, vacua in the \mathcal{J}_t polarization are related to those in the $\mathcal{J}_{t'}$ polarization by the action of any $g \in G_{\mathbb{C}}$ which maps t to t' in \mathcal{M} . Now recall that $|\Omega\rangle_0$ is a vacuum in the polarization at $t = 0$ and that T_i act by translations on \mathcal{M}_D , so $\exp(t^i T_i)$ maps $t = 0$ to t . With this in mind, we write

$$|\Omega\rangle_t = \exp(t^i T_i) |\Omega\rangle_0. \tag{3.22}$$

Since $|\Omega\rangle_0$ is annihilated by the special conformal generators S_i and transforms irreducibly under the “Lorentz” generators R_i with definite scale dimension under D , one can think of Ψ_{hol} as a generalized conformal wave function associated with a unitary representation of G in the sense of [51].

One could ask whether (3.22) actually defines the same vacuum that appears in (3.21). After all, the vacuum is only determined up to rescaling, so *a priori* the choice (3.22) could differ by a function of (t, \bar{t}) from the one that should appear in (3.21). We will verify that (3.22) is the correct choice by showing that

$$\Psi_{\text{hol}}(t^i, y_I) = \langle \Psi | \exp(y_I \widehat{p}^I) \exp(t^i T_i) | \Omega \rangle_0 \tag{3.23}$$

obeys exactly the holomorphic anomaly equations (2.69), (2.70) (which would certainly be spoiled if one multiplied Ψ_{hol} by some non-constant function of (t, \bar{t}) .) Indeed, taking (3.23) as the definition of Ψ_{hol} , the holomorphic anomaly equations follow directly from facts about the Schrödinger-Weil representation of \widetilde{G} , as we now show.

It is convenient to begin by reversing the order of the operators that appear in (3.23). Using $[\widehat{p}^j, T_i] = \delta_i^j \widehat{p}^0$ and $[\widehat{p}^0, T_i] = 0$, (3.23) becomes

$$\Psi_{\text{hol}}(t^i, y_I) = \langle \Psi | \exp(t^i T_i) \exp\left(y_i \widehat{p}^i + (y_0 + t^i y_i) \widehat{p}^0\right) | \Omega \rangle_0. \quad (3.24)$$

So setting $w = y_0 + t^i y_i$,

$$\Psi_{\text{hol}}(t^i, y_i, w) = \langle \Psi | \exp(t^i T_i) \exp\left(y_i \widehat{p}^i + w \widehat{p}^0\right) | \Omega \rangle_0. \quad (3.25)$$

Note that this shift from y_0 to w is exactly the one made in (2.73), which converted the anomaly equations (2.69), (2.70) to their shifted variants (2.74), (2.75). We will now verify that (3.25) satisfies (2.74), (2.75), which we reproduce for convenience:

$$\left[\frac{\partial}{\partial t^i} - \frac{i}{2} C_{ijk} \frac{\partial^2}{\partial y_j \partial y_k} + y_i \frac{\partial}{\partial w} \right] \Psi_{\text{hol}} = 0, \quad (3.26)$$

$$\frac{\partial}{\partial t^i} \Psi_{\text{hol}} = 0. \quad (3.27)$$

The second anomaly equation (3.27) just says Ψ_{hol} is holomorphic in t^i ; this is manifest from (3.23), since $\widetilde{G}_{\mathbb{C}}$ acts holomorphically. To establish the first anomaly equation (3.26) we use the operator equation (3.20) which holds in the Schrödinger-Weil representation. Inserting this operator into the right side of (3.24) gives

$$\langle \Psi | \exp(t^j T_j) \left(Z T_i + \frac{1}{2} C_{ijk} \widehat{p}^j \widehat{p}^k - \widehat{p}^0 \widehat{q}_i \right) \exp\left(y_j \widehat{p}^j + w \widehat{p}^0\right) | \Omega \rangle_0 = 0. \quad (3.28)$$

The \widehat{q}_i operator annihilates $|\Omega\rangle_0$; commuting it through $\exp y_i \widehat{p}^i$ gives a commutator $[\widehat{q}_i, y_i \widehat{p}^i] = -y_i Z$. Also, recall that Z acts as the scalar $i\hbar = i$. Altogether (3.28) becomes

$$\langle \Psi | \exp(t^j T_j) \left(iT_i + \frac{1}{2} C_{ijk} \widehat{p}^j \widehat{p}^k + i \widehat{p}^0 y_i \right) \exp\left(y_j \widehat{p}^j + w \widehat{p}^0\right) | \Omega \rangle_0 = 0, \quad (3.29)$$

which is simply

$$\left[i \frac{\partial}{\partial t^i} + \frac{1}{2} C_{ijk} \frac{\partial^2}{\partial y_j \partial y_k} + i y_i \frac{\partial}{\partial w} \right] \Psi_{\text{hol}} = 0. \quad (3.30)$$

This is our desired anomaly equation (3.26).

3.6 Geometry of the matrix elements

In the last section we realized Ψ_{hol} as a matrix element for the group \widetilde{G} , or more precisely its complexification $\widetilde{G}_{\mathbb{C}}$. We now make a few extra comments about this construction, to elucidate its geometric nature. In particular, we will explain how the metaplectic correction we discussed in section 2.3 arises here as a “zero point energy” of the vacuum $|\Omega\rangle_0$. We will also see that the wave function is naturally considered as a section of a prequantum line bundle, as usual in geometric quantization.

We begin by noting that our construction (3.23) has the general form

$$\Psi_{\text{hol}}(t, y) = \langle \Psi | g_{t,y} | \Omega \rangle_0 \quad (3.31)$$

where $g_{t,y}$ is an element in $\tilde{G}_{\mathbb{C}}$ depending holomorphically on (t, y) . It is natural then to consider Ψ_{hol} simply as a function on $\tilde{G}_{\mathbb{C}}$,

$$\Psi_{\text{hol}}(g) = \langle \Psi | g | \Omega \rangle_0. \quad (3.32)$$

One might think that this function contains more information than did $\Psi_{\text{hol}}(t, y)$, since the number of variables in g is more than in (t, y) . However, $\Psi_{\text{hol}}(g)$ obeys some extra constraints, arising from the fact that $|\Omega\rangle_0$ is an eigenvector for some of the generators of $\tilde{G}_{\mathbb{C}}$. First, $|\Omega\rangle_0$ is annihilated by \hat{q}_I , S^i , and D_j^i ; letting \tilde{P} denote the complex group they generate, this implies immediately that

$$\Psi_{\text{hol}}(gp) = \Psi_{\text{hol}}(g), \quad p \in \tilde{P}. \quad (3.33)$$

Hence Ψ_{hol} can be considered as a function on $\tilde{G}_{\mathbb{C}}/\tilde{P}$. This agrees well with the interpretation of Ψ_{hol} as a ‘‘conformal wave function.’’ Moreover, under the dilatation generator D , $|\Omega\rangle_0$ has eigenvalue $\frac{1}{2}(n_v + 3)$, and under the central generator Z of H , $|\Omega\rangle_0$ has eigenvalue $i\hbar = i$. This implies that

$$\Psi_{\text{hol}}(g \exp(\lambda D + \zeta Z)p) = \exp\left(\frac{1}{2}(n_v + 3)\lambda + i\zeta\right) \Psi_{\text{hol}}(g), \quad \lambda \in \mathbb{C}, \zeta \in \mathbb{C}, p \in \tilde{P}. \quad (3.34)$$

Thus one can also divide out by the larger complex group \tilde{Q} generated by \tilde{P} , D and Z , at the cost that we now describe $\Psi_{\text{hol}}(g)$ as a holomorphic section of a holomorphic line bundle over $\tilde{G}_{\mathbb{C}}/\tilde{Q}$, rather than as an honest function. This line bundle is constructed in a tautological way: the state $|\Omega\rangle_0$ transforms in a 1-dimensional representation of \tilde{Q} ; call this representation ρ , and then define

$$\mathcal{V} = \left\{ (g, z) \in \tilde{G}_{\mathbb{C}} \times \mathbb{C} \right\} / \left[(g, z) \sim (gq, \rho(q)z) : q \in \tilde{Q} \right]. \quad (3.35)$$

Any Ψ_{hol} satisfying (3.34) gives a section of \mathcal{V} . So our construction has naturally led to a section Ψ_{hol} of the line bundle \mathcal{V} over $\tilde{G}_{\mathbb{C}}/\tilde{Q}$.

What is the meaning of this quotient of complex groups? Recall from section 3.2 that in defining the moduli space \mathcal{M} we first considered a larger space \mathcal{M}_D which was a homogeneous space for $G_{\mathbb{C}}$, and then obtained \mathcal{M} as an open G -orbit inside \mathcal{M}_D . Here we have done something precisely analogous, now for the extended group \tilde{G} instead of G : letting $\tilde{\mathcal{M}}_D$ denote $\tilde{G}_{\mathbb{C}}/\tilde{Q}$, we can construct $\tilde{\mathcal{M}}$ as an open orbit for the real group \tilde{G} . This $\tilde{\mathcal{M}}$ is the exact counterpart of the $\tilde{\mathcal{M}}$ which appeared in section 2.2: it is a vector bundle over $\mathcal{M} = G/K$, with phase-space fibers generated by the action of H . It is equipped with a natural complex structure inherited from $\tilde{\mathcal{M}}_D$, which varies along the fibers, and also inherits the holomorphic line bundle \mathcal{V} , of which Ψ_{hol} is a section. It can also be written as a quotient $\tilde{\mathcal{M}} = \tilde{G}/\tilde{K}$, where \tilde{K} is the group generated by $K \subset G$ and the Heisenberg center Z (so \tilde{K} is non-compact).

How should we interpret the fact that Ψ_{hol} comes out as a section of the holomorphic line bundle \mathcal{V} ? In fact, this is exactly what one expects from the point of view of geometric quantization: the wave functions are sections of the ‘‘prequantum line bundle’’ over the

phase space. The fact that the central generator Z of the Heisenberg algebra acts nontrivially means that the restriction of $\Psi_{\text{hol}}(t, y)$ to any fixed $t \in \mathcal{M}$ is naturally a section of a line bundle over the phase space. Of course, since the phase space here is just a linear space, one can suppress the issue by trivializing the line bundle, which is what was done in section 2. Nevertheless, it is gratifying to see the line bundle coming out naturally here.

Similarly, the fact that the dilatation generator $D \in \mathfrak{g}$ acts nontrivially means that even after fixing $y = 0$, $\Psi_{\text{hol}}(t)$ is still a section of a homogeneous line bundle over \mathcal{M} , characterized by the D -eigenvalue $\frac{1}{2}(n_v + 3)$. This line bundle is exactly the bundle $\mathcal{K}^{-\frac{1}{2}}$ related to the metaplectic correction to geometric quantization, which we discussed in section 2.3. To see this, it suffices to note that a holomorphic top-form on V is an element of $\wedge^{\text{top}}(V_+^*)$, and V_+^* has n_v basis elements with weight 1 and one basis element with weight 3 under $\text{Ad}(g(t))D$, so their wedge product has total weight $n_v + 3$.

To sum up: We have seen that starting from a state $|\Psi\rangle$ in the Schrödinger-Weil representation of \tilde{G} one can canonically construct a matrix element $\langle \Psi|g|\Omega\rangle$, which is naturally considered as a section of a holomorphic line bundle \mathcal{V} over a coset space \tilde{G}/\tilde{K} . This \mathcal{V} is the line bundle in which a wave function should live: it incorporates the prequantum line bundle and the metaplectic correction. The formula (3.23) can be understood as defining a particular trivialization of \mathcal{V} , in which the holomorphic anomaly equations of section 2.6 hold.

3.7 Modularity

It is natural to ask what happens if $|\Psi\rangle$ is invariant under some discrete group $\Gamma \subset G$. The line bundle \mathcal{V} over \tilde{G}/\tilde{K} defined in the last section has an obvious action of G . So we can define a quotient bundle over the double coset space $\Gamma\backslash\tilde{G}/\tilde{K}$, and the matrix element $\langle \Psi|g|\Omega\rangle$ is then a holomorphic section of that quotient. This is the usual situation for holomorphic modular forms: they can be understood as sections of holomorphic line bundles over double coset spaces. The double coset space $\Gamma\backslash\tilde{G}/\tilde{K}$ of interest here is a vector bundle over $\Gamma\backslash G/K = \Gamma\backslash\mathcal{M}$; each fiber is a copy of the linear phase space.

If we work with a specific coordinate system on \tilde{G} and trivialization of \mathcal{V} , such as the ones used in (3.23) to make contact with section 2, then $\langle \Psi|g|\Omega\rangle$ will be represented by a holomorphic function, and the Γ -invariance will imply some modular transformation law for this function. Starting from (3.23), for example, one can derive the transformation under $\gamma \in \Gamma$ by considering

$$\Psi_{\text{hol}}(t^i, y_I) = \langle \Psi|\gamma \exp(y_I \hat{p}^I) \exp(t^i T_i) |\Omega\rangle_0 \tag{3.36}$$

and then commuting γ to the right. The result of this procedure is the same modular transformation law we found in section 2.8 for the topological string partition function. In other words, the condition that the string theory is invariant under Γ acting by biholomorphisms on the Calabi-Yau threefold is equivalent to the condition that the state $|\Psi\rangle$ is invariant under Γ acting in the Schrödinger-Weil representation.

One could go further and ask what happens if Γ also includes a discrete subgroup of H , for example, the subgroup generated by $\exp 2\pi i \hat{p}$ and $\exp 2\pi i \hat{q}$. In this case the phase-space fibers of $\Gamma\backslash\tilde{G}/\tilde{K}$ become tori, varying in a family over the moduli space. Formally

then, our construction would lead to a theta function, a holomorphic section of \mathcal{V} over this family of tori. However, this cannot be quite right: \mathcal{V} actually has no holomorphic sections over the torus, because its curvature is not positive definite but rather has one negative eigenvalue; this is a manifestation of the problem we mentioned at the end of section 2.3.¹⁴ From the general theory of geometric quantization one can guess what kind of “theta function” one could hope for: it should be an element in the sheaf cohomology H^1 of \mathcal{V} rather than H^0 . Such theta functions have not been discussed very much in the literature, but see *e.g.* [52, 53]. It would be interesting to understand the relation between our formal construction (which seems to agree with the topological string) and this sheaf cohomology. Of course, *a priori* there is no reason why the topological string should be invariant under any subgroup of H ; we discuss this further in section 4.

3.8 Fourier-Jacobi coefficients and the minimal representation

To this point, there has been a perfect analogy between the topological amplitude on a Hermitian symmetric tube domain and a Jacobi theta series: in either case, the function satisfies a holomorphic heat equation ((1.1) or (2.74)), and can be written as a matrix element in the Schrödinger-Weil representation of a non-reductive group $\tilde{G} = G \ltimes H$, as

$$\Psi(g) = \langle \Psi | g | \Omega \rangle_0. \tag{3.37}$$

Moreover, in each case the heat equations obeyed by $\Psi(g)$ follow directly from operator equations (1.7), (3.20) which hold in the Schrödinger-Weil representation.

It is natural to ask whether this analogy can be pursued further. As discussed in the introduction, the Schrödinger-Weil representation of $SL(2, \mathbb{R}) \ltimes H$ occurs inside the metaplectic representation of a larger group $Sp(4, \mathbb{R})$, the annihilator of which contains the operator equation (1.7). Can one similarly obtain the Schrödinger-Weil representation of \tilde{G} and the operator equation (3.20) from a representation of a larger reductive group G' ?

There is a natural candidate for this G' which is well motivated from physics: upon reducing $\mathcal{N} = 2, d = 4$ supergravity on a circle to three dimensions, the vector multiplet moduli space \mathcal{M} is enlarged to a quaternionic-Kähler space known as the “c-map” of \mathcal{M} . When \mathcal{M} is a Hermitian symmetric tube domain G/K of complex dimension n_v , its c-map is a quaternionic-Kähler symmetric space $G'/(G^c \times SU(2))$, of real dimension $4n_v + 4$. Here G^c is a compact real form of G . We tabulate the groups in question in Table 3.

Just as G was the “conformal group” associated to a Jordan algebra J , the group G' can be described as the “quasi-conformal group” associated to the same J [26, 28]. The reason for this terminology is that G' can be characterized as the invariance group of the “quartic light-cone” in $V_{\mathbb{C}}$,

$$I_4(p^I - \bar{p}^I, q_I - \bar{q}_I) + 2(k - \bar{k} + p^I \bar{q}_I - \bar{p}^I q_I)^2 = 0, \tag{3.38}$$

¹⁴One might ask where exactly we went wrong. Presumably the issue is that in constructing Ψ_{hol} by the expression (3.23) we are trying to analytically continue the matrix elements of the Schrödinger-Weil representation beyond their actual domain of analyticity. A symptom of this is that the states $|\Omega\rangle_t$ which we use are badly non-normalizable.

G	K	G'	K'	n_v
$SU(1,1) \times SO(2,n)$	$U(1) \times U(1) \times SO(n)$	$SO(n+2,4)$	$SU(2) \times SU(2) \times SO(n+2)$	$n+1$
$SL(2, \mathbb{R})$	$U(1)$	$G_{2(2)}$	$SU(2) \times SU(2)$	1
$Sp(6, \mathbb{R})$	$U(1) \times SU(3)$	$F_{4(4)}$	$SU(2) \times Sp(6)$	6
$SU(3,3)$	$U(1) \times SU(3) \times SU(3)$	$E_{6(2)}$	$SU(2) \times SU(6)$	9
$SO^*(12)$	$U(1) \times SU(6)$	$E_{7(-5)}$	$SU(2) \times SO(12)$	15
$E_{7(-25)}$	$U(1) \times E_6$	$E_{8(-24)}$	$SU(2) \times E_7$	27

Table 3: Isometry groups G' of the moduli spaces occurring upon dimensional reduction from 4 to 3 dimensions, with their maximal compact subgroups K' , and the corresponding G , K from 4 dimensions. All groups are given only up to finite covering or finite quotient.

	-3	-2	-1	0	1	2	3
2				Z			
1	$i\hat{q}_0$		$i\hat{q}_i$		$i\hat{p}^i$		$i\hat{p}^0$
0		S^i		D_j^i, D, Δ		T_i	
-1	$i\hat{Q}^0$		$i\hat{Q}^i$		$i\hat{P}_i$		$i\hat{P}_0$
-2				\tilde{Z}			

Table 4: A list of the generators of the Lie algebra \mathfrak{g}' , with two gradings, by D (horizontal) and by Δ (vertical). This extends figure 2, which was the analogous picture for $\tilde{\mathfrak{g}} \subset \mathfrak{g}'$.

more precisely, G' is the group leaving the left side of (3.38) invariant up to a multiplication by a function of (p^I, q_I, k) times a function of $(\bar{p}^I, \bar{q}_I, \bar{k})$. The twistor space of the quaternionic-Kähler space $G'/G^c \times SU(2)$ admits a Einstein-Kähler metric whose Kähler potential can be shown to be proportional to the logarithm of (3.38), making the symmetry under G' manifest [54].

The Lie algebra \mathfrak{g}' has a bigrading extending those of \mathfrak{g} and $\tilde{\mathfrak{g}}$, with a corresponding picture extending Figure 2, given in Figure 4. Note in particular that Z is not a central generator in \mathfrak{g}' : together with \tilde{Z} and Δ it generates an $\mathfrak{sl}(2, \mathbb{R})$ subalgebra of \mathfrak{g}' . The group \tilde{G} is then the centralizer of Z inside G' . This implies that if one begins with a representation \mathcal{H}' of G' and then reduces \mathcal{H}' by fixing Z to some real value \hbar ,¹⁵ the resulting space \mathcal{H}_\hbar will be a representation of \tilde{G} .

Having found the candidate G' , it remains to exhibit some \mathcal{H}' such that the corresponding \mathcal{H}_\hbar is the Schrödinger-Weil representation of \tilde{G} . It turns out that the desired \mathcal{H}' is an analogue of the metaplectic representation of $Sp(4, \mathbb{R})$, known as the *minimal* representation of G' . These representations have been studied extensively in the mathematics literature, *e.g.* [55–62], and also by physicists, *e.g.* [26–28, 63–69]. The action of the generators of \mathfrak{g}' on functions in $n_v + 2$ variables has been given explicitly in [28, 67, 69] for simple groups in their quaternionic real form, which is the case presently relevant for us; however, the polarization chosen there was not the same as the real polarization we used

¹⁵To be precise, one defines \mathcal{H}_\hbar as a “space of coinvariants,” $\mathcal{H}_\hbar = \mathcal{H}/\{Z\psi - i\hbar\psi : \psi \in \mathcal{H}\}$.

in describing the Schrödinger-Weil representation above, making a direct comparison difficult. Now we sketch a realization of the minimal representation which makes the relation more transparent. The representation space \mathcal{H}' consists of functions $f(p^I, y)$ in $n_v + 2$ variables. The generators of $\tilde{\mathfrak{g}} \subset \mathfrak{g}'$ are just rescaled versions of those in the Schrödinger-Weil representation:

$$Z \mapsto iy^2, \tag{3.39}$$

$$i\hat{q}_0 \mapsto y \frac{\partial}{\partial p^0}, \quad i\hat{q}_i \mapsto y \frac{\partial}{\partial p^i}, \quad i\hat{p}^i \mapsto iyp^i, \quad i\hat{p}^0 \mapsto iyp^0, \tag{3.40}$$

$$S^i \mapsto -\frac{i}{2} C^{ijk} \frac{\partial^2}{\partial p^j \partial p^k} + p^i \frac{\partial}{\partial p^0}, \quad T_i \mapsto +\frac{i}{2} C_{ijk} p^j p^k - p^0 \frac{\partial}{\partial p^i}, \tag{3.41}$$

$$R_i^j \mapsto \delta_i^j p^0 \frac{\partial}{\partial p^0} - p^j \frac{\partial}{\partial p^i} + \frac{1}{2} C_{ikl} C^{jnl} \left(p^k \frac{\partial}{\partial p^n} + \frac{\partial}{\partial p^n} p^k \right). \tag{3.42}$$

In particular, fixing $y = 1$ we clearly recover the Schrödinger-Weil representation of \tilde{G} with $\hbar = 1$; more generally one could fix y to any value, giving the Schrödinger-Weil representation with $\hbar = y^2$. The other generators of \mathfrak{g}' can also be described explicitly. The generator $\Delta \in \mathfrak{g}'$ simply counts the degree in y ,

$$\Delta \mapsto y \frac{\partial}{\partial y} + \frac{1}{2}. \tag{3.43}$$

To obtain the rest of \mathfrak{g}' it is enough to give the action of \tilde{Z} , since $i\hat{Q}_I = [\tilde{Z}, i\hat{q}_I]$ and $i\hat{P}_I = [\tilde{Z}, i\hat{p}_I]$; this action can be given in terms of the quartic invariant,

$$\tilde{Z} \mapsto \frac{1}{2} \frac{\partial^2}{\partial y^2} - \frac{1}{4y^2} \left(I_4(\hat{p}^I, \hat{q}_I) + \kappa \right), \tag{3.44}$$

where κ depends on the choice of ordering one makes in I_4 .

Since we have seen that the topological string state $|\Psi\rangle$ lives in the Schrödinger-Weil representation $\mathcal{H}_{\hbar=1}$ of \tilde{G} , the existence of this extension of the Schrödinger-Weil representation suggests that there could also exist a one-parameter generalization of the topological string. We will speculate on its physical meaning in the next section.

So far we have been discussing the representation theory of the real groups \tilde{G} and G' without regard to their integral subgroups. But this discussion has an automorphic counterpart, which in our language amounts to studying not just the relation between Hilbert spaces \mathcal{H}_{\hbar} and \mathcal{H}' , but also some special states which are invariant under discrete subgroups. Such states give rise to automorphic forms for the corresponding groups. The correspondence is easily described in the case of $G' = Sp(4, \mathbb{R})$: given a holomorphic Siegel modular form $\Theta(\tau, \rho, z)$ for $Sp(4, \mathbb{R})$ one can obtain holomorphic Jacobi forms for $SL(2, \mathbb{R}) \times H$ by Fourier expanding Θ in one of its three complex variables,

$$\Theta(t, \rho, z) = \sum_{m=1}^{\infty} \theta_m(t, z) e^{im\rho}. \tag{3.45}$$

Each $\theta_m(t, z)$ is then a Jacobi form of index m . Now how about our case, where $Sp(4, \mathbb{R})$ is replaced by G' ? The minimal representation of G' is in some cases known to be automorphic, so there is a natural candidate automorphic form Θ , see *e.g.* [61, 70, 71]. The

automorphic extension of our discussion above would involve expanding Θ in e^{\hbar} to obtain Jacobi forms associated to the Schrödinger-Weil representation of \tilde{G} . As far as we know such an expansion is not in the mathematics literature, although some of the necessary ingredients have been studied in [72], which also discusses similar expansions for other “small” representations of G' .

4. Discussion

In this paper, we have shown that the holomorphic anomaly equations for the topological string amplitude can be rephrased as a heat equation for a purely holomorphic section,

$$\left[\frac{\partial}{\partial t^i} - \frac{i}{2} C_{ijk} \frac{\partial^2}{\partial y_j \partial y_k} + y_i \frac{\partial}{\partial w} \right] \Psi_{\text{hol}}(t; w, y^i) = 0. \tag{4.1}$$

Moreover, in the case where the moduli space \mathcal{M} is a Hermitian symmetric tube domain G/K , we have shown that the general solution can be written as a matrix element

$$\Psi_{\text{hol}}(t; y) = \langle \Psi | \exp(y_i p^{\hat{I}}) \exp(t^i T_i) | \Omega \rangle \tag{4.2}$$

in the Schrödinger-Weil representation of a non-reductive group $\tilde{G} = G \ltimes H$. The holomorphic anomaly equation (4.1) then follows from identities in the annihilator of the Schrödinger-Weil representation, which moreover can be derived by embedding in the minimal representation of a larger reductive group G' .

We close with a few comments and speculations.

An extended topological amplitude. The realization of the holomorphic anomaly equation inside the Schrödinger-Weil representation of \tilde{G} is not surprising in light of the wave function interpretation. In our opinion, though, its natural embedding into G' lends credence to the idea that there should exist a one-parameter generalization of the usual topological string amplitude, attached to the minimal representation of G' . The extra parameter is \hbar , the eigenvalue of Z , which was fixed to $\hbar = 1$ in the Schrödinger-Weil representation we used.

Now, what could the meaning of the extra parameter \hbar be? Here we present two speculations, both relying on the fact that the group G' can be interpreted physically as the isometry group after compactification on a circle to three dimensions. First, note that by T-duality along the circle (exchanging IIA and IIB) followed by decompactification to 4 dimensions, G' can be realized as the isometry group of a hypermultiplet moduli space in 4 dimensions. It is thus natural to conjecture that the “generalized topological string amplitude” should compute the infinite series of higher-derivative corrections \tilde{F}_g on this hypermultiplet moduli space. The extra parameter \hbar would then be related to the presence of the “universal hypermultiplet” containing the physical string coupling, controlling the strength of the spacetime instanton corrections. To make this proposal sharper it would be useful to have an off-shell superspace description of the hypermultiplet derivative couplings, perhaps along the lines of [73].

For the other speculation we need to discuss how the Heisenberg generators in \tilde{G} are realized in G' . They occur as shift symmetries for new scalars that come from the

dimensional reduction from 4 to 3 dimensions: in particular, the central generator Z acts by shifting the scalar which arises from dualization of the $d = 3$ gauge field $g_{\mu t}$. This scalar can be thought of as a dual potential for the Taub-NUT charge, in which case fixing Z is equivalent to fixing the NUT charge of the 4-dimensional spacetime. As was exploited in [17, 74], T-dualizing the space with \hbar units of NUT charge leads to \hbar NS5-brane instantons wrapped on X [75]. In particular, suppose we start with the IIA theory; then we wind up with \hbar NS5-brane instantons of Type IIB. In [74, 76] the $\hbar = 1$ case of this duality was used to argue that the partition function of the single NS5-brane instanton captures the A model partition function on X ; this gave a physical interpretation to the results of [77–79], which argued that the A model partition function can be computed using the maximally supersymmetric $U(1)$ gauge theory on X , mathematically interpreted as the $U(1)$ “Donaldson-Thomas” theory [80]. This suggests that our “generalized topological amplitude” should include not only the $U(1)$ gauge theory but also the higher $U(\hbar)$ theories for all \hbar ; their partition functions should be organized into a single vector in the minimal representation of G' .

While these two proposals are both speculative, at least they may be consistent with one another: it is expected that the hypermultiplet moduli space should be corrected by NS5-brane instantons, so to say that the generalized amplitude computes corrections to the hypermultiplet moduli space is not unrelated to saying it computes the partition function of NS5-brane instantons. Moreover, they are broadly consistent with the OSV conjecture, since the instantons correcting the three-dimensional vector-multiplet moduli space are four-dimensional black holes running along the thermal circle, T-dual to the NS5-brane instantons. We are hopeful that a proper understanding of the meaning of the extended topological amplitude will lead to an exact form of the OSV conjecture, valid at finite values of the charges, along the lines of the ideas in [81, 11].

Quantizing the intermediate Jacobian. One of the main themes of this paper has been the analogy between the topological string partition function Ψ_{hol} and the Jacobi theta series $\theta(\tau, z)$. There is one crucial property of $\theta(\tau, z)$ which we have mostly overlooked: it transforms simply under shifts $z \rightarrow z + 1$, $z \rightarrow z + \tau$. Geometrically, these transformations reflect the fact that θ arises not from quantization of the linear space $H^1(T^2, \mathbb{R})$ but from quantization of the Jacobian torus, $H^1(T^2, \mathbb{R})/H^1(T^2, \mathbb{Z})$. Equivalently, one can continue to think in terms of the linear space, but require invariance under the integer form of the Heisenberg group. This is a very strong constraint on the state, determining it up to at most a finite-dimensional ambiguity.

From the point of view of the worldsheet of the topological string theory, there is no obvious reason why the topological string state should have such an invariance. On the other hand, if one takes seriously the embedding $\tilde{G} \subset G'$ we advocated above, one finds that the phase space variables in the topological string get a physical interpretation: they are the Wilson lines of the $n_v + 1$ gauge fields around the $d = 4 \rightarrow d = 3$ compactification circle. As such, they are naturally circle-valued, and one might expect that any partition function which depends on them will have to be periodic in an appropriate sense. One would then be led to consider quantization of the intermediate Jacobian $J = H^3(X, \mathbb{R})/H^3(X, \mathbb{Z})$, with

the corresponding theta functions as candidates for the full topological string partition function.¹⁶

One might well be skeptical of whether such a simple and universal object could really capture all the complexity of the topological string theory on Calabi-Yau threefolds. Moreover, even if the topological string state indeed turned out to be invariant under shifts by $H^3(X, \mathbb{Z})$, there would still be the problem of extracting the perturbative expansion. This is not as trivial as it sounds: as we discussed in section 3.7, in the complex structure we are using, the geometric description of such a theta function would be as a class in sheaf cohomology H^1 rather than as a function. Said differently, the series defining such a theta function is badly divergent. Still, the case of a Calabi-Yau one-fold gives some ground for optimism: the partition function which counts covering maps to the torus can indeed be expressed in terms of a “generalized Jacobi form,” defined by a formal product which definite modular properties [83, 84]; in particular, the $SL(2, \mathbb{Z})$ transformation of the variables seems to have a natural interpretation in terms of quantization of a phase space in the 4-dimensional representation of $SL(2, \mathbb{R})$.

Extensions to $\mathcal{N} = 4$ and $\mathcal{N} = 8$. In this paper we have been concerned only with the case of $\mathcal{N} = 2$ supersymmetry. Given our focus on cases where the moduli space is a symmetric space, it might have seemed more natural to consider the $\mathcal{N} = 4$, $\mathcal{N} = 6$ and $\mathcal{N} = 8$ theories first, since in those cases the moduli space is always a symmetric space, which we again write as G'/K' in $d = 3$. Then one can construct the minimal representation of G' ; its matrix elements are very special functions on G' , obeying a large number of differential equations, which could be easily written down much as we did above in the $\mathcal{N} = 2$ case. It is natural to wonder whether these equations arise somewhere in physics.

There is an important difference between the $\mathcal{N} = 2$ supergravities defined by cubic Jordan algebras, which we considered above, and the $\mathcal{N} = 8$ theory or the $\mathcal{N} = 4$ theories coupled to vector multiplets. Namely, in general the groups G occurring in the latter cases in $d = 4$ do not have holomorphic discrete series representations. Roughly this means one should not expect to construct holomorphic “wave functions” from the representations of G . So the analogues of the anomaly equations which appear must have a somewhat subtler interpretation. One notable exception is the $\mathcal{N} = 4$ supergravity coupled to 2 vector multiplets, where the moduli space in $d = 4$ is $(SL(2, \mathbb{R}) \times SO(2, 6))/(U(1) \times U(1) \times SO(6))$. The existence of holomorphic discrete series in this case may be related to the fact that this theory can be obtained by consistent truncation from the $\mathcal{N} = 6$ theory, for which the corresponding group $SO^*(12)$ does admit holomorphic discrete series representations.

In the case of Type II string theory compactified on K3 one might ask whether the differential equations one obtains are related to the holomorphic anomaly of the $\mathcal{N} = 4$ topological string on K3 [85, 86]. To obtain such a relation one would presumably have

¹⁶This intermediate Jacobian has appeared in [82], where it was argued that, if one wraps an M-theory fivebrane on X , the dependence of its partition function on a flat background C field is given by a theta function similar to those we are discussing. A similar argument was made in [17] for the “classical part” of the partition function of a stack of NS fivebranes in Type IIA.

to decompactify from $d = 3$ to $d = 6$, which should correspond to fixing 3 generators of G' (analogous to the $\mathcal{N} = 2$ case where we decompactified from $d = 3$ to $d = 4$ and fixed the single generator Z .) The centralizer of these generators in $G' = O(24, 8)$ should be some nilpotent extension $\tilde{G} = O(20, 4) \times N$, where $O(20, 4)$ arises as the isometry group of the moduli space in $d = 6$. Then the natural conjecture, analogous to what we found in the $\mathcal{N} = 2$ case, is that the anomaly equations of the $\mathcal{N} = 4$ topological string can be interpreted as statements about some representation of this \tilde{G} .

One can similarly ask whether any meaning can be attached to the differential equations which arise in the $\mathcal{N} = 8$ case from the minimal representation of $E_{8(8)}$.

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A. Holomorphic ambiguities and real polarization

In this appendix, we discuss the relation between the “holomorphic ambiguities” found by BCOV in their procedure for solving the holomorphic anomaly equations recursively and the real polarized topological amplitude defined in (A.9).

In [6, section 6.2], BCOV introduce a variant of (2.9),

$$\tilde{\Psi}_{\text{BCOV}}(t^i, \bar{t}^{\bar{j}}; \tilde{x}^i, \varphi; \tilde{\lambda}) = \tilde{\lambda}^{1-\frac{\chi}{24}} \exp \left[W \left(t^i, \bar{t}^{\bar{j}}; \frac{\tilde{x}^i}{1-\varphi}, \frac{\tilde{\lambda}}{1-\varphi} \right) \right]. \quad (\text{A.1})$$

Here $\tilde{\lambda}$ and φ are redundant variables: φ describes variations of the inverse topological coupling λ^{-1} , while $\tilde{\lambda}$ only serves to organize the perturbative series. Then the first anomaly equation (2.10) becomes

$$\partial_{\bar{t}^i} = \frac{\tilde{\lambda}^2}{2} e^{2K} \bar{C}_{\bar{i}\bar{j}\bar{k}} g^{\bar{j}\bar{j}} g^{k\bar{k}} \frac{\partial^2}{\partial \tilde{x}^{\bar{j}} \partial \tilde{x}^{\bar{k}}} - g_{\bar{i}j} \tilde{x}^j \frac{\partial}{\partial \varphi} \quad (\text{A.2})$$

Moreover, BCOV observe that the “conjugate” equation

$$\partial_{\bar{t}^i} = -\frac{\tilde{\lambda}^2}{2} e^{2K} \bar{C}_{\bar{i}\bar{j}\bar{k}} g^{\bar{j}\bar{j}} g^{k\bar{k}} \frac{\partial^2}{\partial \tilde{x}^{\bar{j}} \partial \tilde{x}^{\bar{k}}} - g_{\bar{i}j} \tilde{x}^j \frac{\partial}{\partial \varphi} \quad (\text{A.3})$$

is solved by

$$\tilde{\Psi}_Y(t, \bar{t}; \tilde{\lambda}, \tilde{x}, \varphi) = \exp \left[-\frac{1}{2\tilde{\lambda}^2} (\Delta_{ij} \tilde{x}^i \tilde{x}^j + 2\Delta_{i\varphi} \tilde{x}^i \varphi + \Delta_{\varphi\varphi} \varphi^2) + \frac{1}{2} \log \left(\frac{\det \Delta}{\tilde{\lambda}^2} \right) \right] \quad (\text{A.4})$$

where the “inverse propagator” Δ is defined to obey the equations

$$\Delta = \begin{pmatrix} \Delta_{\varphi\varphi} & \Delta_{i\varphi} \\ \Delta_{i\varphi} & \Delta_{ij} \end{pmatrix}, \quad \Delta^{-1} = \begin{pmatrix} -2S & -S^i \\ -S^i & -S^{ij} \end{pmatrix} = \begin{pmatrix} -2S & -g^{i\bar{j}} \partial_{\bar{i}} S \\ -g^{i\bar{j}} \partial_{\bar{j}} S & -g^{i\bar{j}} \partial_{\bar{j}} (g^{j\bar{k}} \partial_{\bar{k}} S) \end{pmatrix}, \quad (\text{A.5})$$

and S is a local section of \mathcal{L}^{-2} whose third derivative reproduces the tree-level three-point function,

$$C_{i\bar{j}\bar{k}} = e^{-2K} D_{\bar{i}} D_{\bar{j}} D_{\bar{k}} S. \quad (\text{A.6})$$

Note that this does not define S unambiguously. Using (A.2), (A.3) BCOV conclude that the integral

$$Z(t, \bar{t}) = \int d\tilde{x} d\varphi \tilde{\Psi}_{\text{BCOV}}(t, \bar{t}; \tilde{x}, \tilde{\lambda}, \varphi) \Psi_Y(t, \bar{t}; \tilde{x}, \tilde{\lambda}, \varphi) \quad (\text{A.7})$$

is in fact independent of \bar{t} , and identify it as the generating function of the “holomorphic ambiguities” which arise at each genus upon integrating the first anomaly equation.

Now let us relate this construction to the intertwiner from Verlinde’s Ψ_V to the real polarized $\Psi_{\mathbb{R}}$. The first thing to notice is that Eq. (A.2) is identical to (2.15) after shifting $\varphi \rightarrow \varphi + 1$ in (A.1), setting $\tilde{\lambda} = 1$ and identifying $\varphi = -\lambda^{-1}$. Next, we invert the formula (2.57) which gave the Bogoliubov transformation from $\Psi_{\mathbb{R}}$ to Ψ_V , obtaining

$$\Psi_{\mathbb{R}}(p^I) = \int dx^I d\bar{x}^I \sqrt{\det[\text{Im } \tau]} e^{-\frac{1}{2} x^I [\text{Im } \tau]_{IJ} \bar{x}^J} \Psi_V^*(p^I; X^I, \bar{X}^I; \bar{x}^I) \Psi_V(X^I, \bar{X}^I; x^I). \quad (\text{A.8})$$

The integral over \bar{x}^I is easily evaluated (at least formally), leading to

$$\begin{aligned} \Psi_{\mathbb{R}}(p^I) &= \int dx^I \sqrt{\det[\text{Im } \tau]} \exp \left[\frac{i}{2} p^I \bar{\tau}_{IJ} p^J - p^I [\text{Im } \tau]_{IJ} x^J + \frac{1}{4} x^I [\text{Im } \tau]_{IJ} x^J \right] \\ &\times \Psi_V(X^I, \bar{X}^I; x^I). \end{aligned} \quad (\text{A.9})$$

Setting $p^I = 0$ in this expression, and replacing the large phase space variables x^I in terms of (x^i, λ) in the exponent, we find

$$\begin{aligned} \Psi_{\mathbb{R}}(p^I) &= \int d\lambda^{-1} dx^i \Psi_V(X^I, \bar{X}^I; x^i, \lambda) \sqrt{\det[\text{Im } \tau]} \\ &\exp \left[\frac{1}{4} \left(\lambda^{-2} X^I [\text{Im } \tau]_{IJ} X^J + 2\lambda^{-1} e^{-K/2} x^i f_i^I [\text{Im } \tau]_{IJ} X^J + e^{-K} x^i f_i^I [\text{Im } \tau]_{IJ} f_j^J x^j \right) \right]. \end{aligned} \quad (\text{A.10})$$

This coincides with (A.4) if we identify

$$\Delta_{ij} = -2e^{-K} f_i^I [\text{Im } \tau]_{IJ} f_j^J, \quad (\text{A.11})$$

$$\Delta_{i\varphi} = 2e^{-K/2} f_i^I [\text{Im } \tau]_{IJ} X^J, \quad (\text{A.12})$$

$$\Delta_{\varphi\varphi} = -2X^I [\text{Im } \tau]_{IJ} X^J, \quad (\text{A.13})$$

where we introduced the standard special geometry notation $f_i^I := e^{K/2} D_i X^I$.

Inverting the matrix Δ with these entries, we obtain

$$S = e^{2K} \bar{X}^I [\text{Im } \tau]_{IJ} \bar{X}^J, \quad (\text{A.14})$$

$$S_{\bar{i}} = 2e^{3K/2} \bar{f}_{\bar{i}}^I [\text{Im } \tau]_{IJ} \bar{X}^J, \quad (\text{A.15})$$

$$S_{\bar{i}\bar{j}} = 2e^K \bar{f}_{\bar{i}}^I [\text{Im } \tau]_{IJ} \bar{f}_{\bar{j}}^J. \quad (\text{A.16})$$

Then using the special geometry identities

$$X^I (\partial_i [\text{Im } \tau]_{IJ}) X^J = f_i^I (\partial_j [\text{Im } \tau]_{IJ}) X^J = 0, \quad f_i^I (\partial_j [\text{Im } \tau]_{IJ}) f_k^J = -\frac{i}{2} e^{-K} C_{ijk}, \quad (\text{A.17})$$

we check that S given by (A.14) indeed satisfies (A.6), and moreover that $S_{\bar{i}} = D_{\bar{i}} S$, $S_{\bar{i}\bar{j}} = D_{\bar{i}} D_{\bar{j}} S$ as required in (A.5).

We conclude that, for the S given in (A.14), $Z(t)$ is proportional to the wave function in the real polarization at $p^I = 0$, namely

$$Z(t) = e^{f_1(t)} \Psi_{\mathbb{R}}(0) \quad (\text{A.18})$$

where the exponential factor arises from the redefinition (2.14). We emphasize that (A.18) holds only for the S given in (A.14), which need not be globally well defined; for the choice of S made in [6], $Z(t)$ in fact contains the infinite series of holomorphic ambiguities.

B. Wave function in the positive-definite polarization

With a little more work, we can also determine the intertwiner Ψ_{W} from the real polarization to a positive definite (“Weil”) polarization given by the Hodge \star : as discussed below (2.30), this Ψ_{W} should be obtained from the intertwiner Ψ_{V} to the indefinite (“Griffiths”) polarization by Fourier transforming from λ^{-1} to a new variable μ . So we begin from the formula (2.60) for Ψ_{V} , replace the large phase space variables x^I with the small phase space ones (x^i, λ) using (2.37), and then formally Fourier transform

$$\Psi_{\text{W}}(p^I; X^I, \bar{X}^I; x^i, \mu) = \int d\lambda^{-1} \exp(\lambda^{-1} \mu) \Psi_{\text{V}}(p^I; X^I, \bar{X}^I; x^i, \lambda). \quad (\text{B.1})$$

We thus obtain Ψ_{W} in terms of the negative-definite period matrix

$$\mathcal{N}_{IJ} = \bar{\tau}_{IJ} + 2i \frac{[\text{Im } \tau]_{IK} X^K [\text{Im } \tau]_{JL} X^L}{X^K [\text{Im } \tau]_{KL} X^L}, \quad (\text{B.2})$$

namely, we find

$$\begin{aligned} \Psi_{\text{W}}(p^I; X^I, \bar{X}^I; x^i, \mu) = & \frac{\sqrt{\det[\text{Im } \tau]}}{\sqrt{X^I [\text{Im } \tau]_{IJ} \bar{X}^J}} \exp \left[\frac{i}{2} p^I \mathcal{N}_{IJ} p^J - 4ie^K p^I [\text{Im } \mathcal{N}]_{IJ} \bar{X}^J \mu \right. \\ & - e^{-K/2} p^I [\text{Im } \mathcal{N}]_{IJ} f_j^J x^j + 4e^{2K} \bar{X}^I [\text{Im } \mathcal{N}]_{IJ} \bar{X}^J \mu^2 \\ & \left. - 2ie^{K/2} f_i^I [\text{Im } \mathcal{N}]_{IJ} \bar{X}^J x^i \mu - \frac{1}{4} e^{-K} f_i^I [\text{Im } \mathcal{N}]_{IJ} f_j^J x^i x^j \right]. \end{aligned} \quad (\text{B.3})$$

Here we used the special geometry properties

$$\bar{X}^I [\text{Im } \mathcal{N}]_{IJ} \bar{X}^J = \frac{1}{4} \frac{e^{-2K}}{X^I [\text{Im } \tau]_{IJ} X^J} \quad (\text{B.4})$$

$$f_i^I [\text{Im } \mathcal{N}]_{IJ} \bar{X}^J = -\frac{1}{2} \frac{e^{-K} f_i^I [\text{Im } \tau]_{IJ} X^J}{X^I [\text{Im } \tau]_{IJ} X^J} \quad (\text{B.5})$$

$$f_i^I [\text{Im } \mathcal{N}]_{IJ} f_j^J = \frac{(f_i^I [\text{Im } \mathcal{N}]_{IK} X^K)(f_j^J [\text{Im } \mathcal{N}]_{JL} X^L)}{X^I [\text{Im } \tau]_{IJ} X^J} - f_i^I [\text{Im } \tau]_{IJ} f_j^J \quad (\text{B.6})$$

$$p^I [\text{Im } \mathcal{N}]_{IJ} \bar{X}^J = -\frac{1}{2} e^{-K} \frac{p^I [\text{Im } \tau]_{IJ} X^J}{X^I [\text{Im } \tau]_{IJ} X^J} \quad (\text{B.7})$$

Defining new large phase space coordinates (for $H^{1,2} \oplus H^{0,3}$) by

$$w^I = 2i \mu \bar{X}^I e^K + \frac{1}{2} x^i D_i X^I, \quad (\text{B.8})$$

the intertwiner between the real and positive-definite polarizations takes the simple form

$$\begin{aligned} \Psi_W(p^I; X^I, \bar{X}^I; w^I) &= \\ &= \frac{\sqrt{\det[\text{Im } \tau]}}{\sqrt{X^I [\text{Im } \tau]_{IJ} X^J}} \exp \left[\frac{i}{2} p^I \mathcal{N}_{IJ} p^J - 2p^I [\text{Im } \mathcal{N}]_{IJ} w^J - w^I [\text{Im } \mathcal{N}]_{IJ} w^J \right] \end{aligned} \quad (\text{B.9})$$

which is closely similar to (2.60).

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